

Lecture Notes: Two-Level Systems in Quantum Optics

(Einstein picture \rightarrow full time-dependent Schrödinger picture \rightarrow Bloch sphere)

Based mainly on Fox (2006), Ch. 4 and Ch. 9.

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1 Why the two-level system is the workhorse model

A remarkably large set of quantum-optical phenomena can be understood by truncating the Hilbert space of an atom (or any quantum emitter) to two relevant stationary states:

$$|1\rangle \equiv \text{lower level}, \quad |2\rangle \equiv \text{upper level}.$$

The model becomes especially powerful when the light field is resonant (or near-resonant) with the transition.

Info: Two-level approximation

We assume only one transition matters:

$$\hbar\omega_0 = E_2 - E_1,$$

and other levels are either far detuned or only contribute indirectly (often modeled as *damping* and *dephasing*).

Logic note (for lecture): How to guide students

Start from the **rate** (Einstein) picture to build intuition about absorption/emission probabilities, then show what it *misses*: the **coherent evolution during the interaction**. The TDSE treatment explains that missing “what happens in between” story.

2 Part I: Standard Einstein picture (rates and coefficients)

2.1 States, populations, and the three radiative processes

Let $N_1(t)$ and $N_2(t)$ be the number of atoms in levels 1 and 2. Consider radiation with spectral energy density $u(\omega)$.

Info: Spectral energy density

$u(\omega)$ has units $\text{J m}^{-3} (\text{rad/s})^{-1}$ and represents the energy per unit volume per unit angular-frequency interval around ω .

There are three processes:

1. **Spontaneous emission:** $|2\rangle \rightarrow |1\rangle$ without requiring incident radiation.
2. **Absorption:** $|1\rangle \rightarrow |2\rangle$ induced by the field near $\omega \approx \omega_0$.
3. **Stimulated emission:** $|2\rangle \rightarrow |1\rangle$ induced by the same field.

2.2 Rate equations (definitions of A and B)

2.2.1 Spontaneous emission

Einstein's A coefficient is defined by

$$\frac{d}{dt}N_2(t) = -A_{21}N_2(t). \quad (1)$$

Solve explicitly:

$$\frac{1}{N_2} \frac{d}{dt}N_2 = -A_{21} \Rightarrow \int_{N_2(0)}^{N_2(t)} \frac{dN_2}{N_2} = - \int_0^t A_{21} dt' \quad (2)$$

$$\Rightarrow \ln\left(\frac{N_2(t)}{N_2(0)}\right) = -A_{21}t \Rightarrow N_2(t) = N_2(0)e^{-A_{21}t}. \quad (3)$$

Define the radiative lifetime

$$\tau \equiv \frac{1}{A_{21}}, \quad (4)$$

so $N_2(t) = N_2(0)e^{-t/\tau}$.

Key point: Meaning of A_{21}

A_{21} sets the **natural decay time scale** of the excited population in the absence of driving.

2.2.2 Absorption

Einstein's B coefficient for absorption is defined by

$$\frac{d}{dt}N_1(t) = -B_{12}^\omega N_1(t) u(\omega). \quad (5)$$

Interpretation: the absorption rate is proportional to (i) how many atoms are available in level 1 and (ii) how much radiation exists at the transition frequency.

2.2.3 Stimulated emission

Stimulated emission is defined similarly by

$$\frac{d}{dt}N_2(t) = -B_{21}^\omega N_2(t) u(\omega). \quad (6)$$

Logic note (for lecture): A conceptual sentence to say aloud

“Spontaneous emission needs no field; absorption and stimulated emission are proportional to the field’s spectral energy density at the transition.”

2.3 Thermal equilibrium and detailed balance: deriving relationships among Einstein coefficients

We now reproduce the classical Einstein argument: place atoms inside a cavity with black-body radiation at temperature T .

Info: Steady state (detailed balance)

At equilibrium, upward transitions (absorption) must balance downward transitions (spontaneous + stimulated emission).

2.3.1 Write the balance equation

Upward rate:

$$R_\uparrow = B_{12}^\omega N_1 u(\omega).$$

Downward rate:

$$R_\downarrow = A_{21}N_2 + B_{21}^\omega N_2 u(\omega).$$

Equilibrium requires

$$B_{12}^\omega N_1 u(\omega) = A_{21}N_2 + B_{21}^\omega N_2 u(\omega). \quad (7)$$

2.3.2 Relate populations via Boltzmann’s law

Thermal equilibrium implies

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} \exp\left(-\frac{\hbar\omega}{k_B T}\right), \quad (8)$$

where g_1, g_2 are degeneracies.

2.3.3 Use Planck’s formula for $u(\omega)$

Black-body radiation has

$$u(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{\exp(\hbar\omega/k_B T) - 1}. \quad (9)$$

2.3.4 Solve for consistency at all T

From (7):

$$B_{12}^\omega N_1 u(\omega) = N_2 (A_{21} + B_{21}^\omega u(\omega)) \quad (10)$$

$$\Rightarrow \frac{N_2}{N_1} = \frac{B_{12}^\omega u(\omega)}{A_{21} + B_{21}^\omega u(\omega)}. \quad (11)$$

Equate (11) with (8) and insert (9). The only way for the expression to match for *all* T is that the coefficients satisfy:

$$g_1 B_{12}^\omega = g_2 B_{21}^\omega, \quad (12)$$

$$A_{21} = \frac{\hbar \omega^3}{\pi^2 c^3} B_{21}^\omega. \quad (13)$$

Key point: Result to remember

Knowing *one* Einstein coefficient determines the others via (12)–(13).

Common pitfall: Units confusion

The B coefficient depends on whether you use $u(\omega)$ or $u(\nu)$ (angular frequency vs ordinary frequency). Conversions introduce factors of 2π .

3 From rates to amplitudes: why the Einstein picture is incomplete

The Einstein picture predicts *transition rates*, but it does **not** describe the coherent evolution of an atom while a resonant light field is present.

Logic note (for lecture): The motivating question

“What happens to the irradiated atom *before* the absorption transition is complete?”

Answer: solve the **time-dependent Schrödinger equation** for a driven two-level system.

4 Part II: Full wavefunction picture (Fox Ch. 9): TDSE and resonant driving

4.1 Set-up: Hamiltonian decomposition and near resonance

We solve

$$\hat{H}\Psi = i\hbar \frac{\partial \Psi}{\partial t}, \quad (14)$$

for a two-level atom with energies E_1, E_2 driven by light of angular frequency ω near the transition:

$$\omega = \omega_0 + \delta\omega, \quad (15)$$

$$\omega_0 = \frac{E_2 - E_1}{\hbar}, \quad |\delta\omega| \ll \omega_0. \quad (16)$$

Split the Hamiltonian into

$$\hat{H} = \hat{H}_0(\mathbf{r}) + \hat{V}(t), \quad (17)$$

where \hat{H}_0 is the atom “in the dark” and $\hat{V}(t)$ is the light–matter interaction.

4.2 Unperturbed eigenstates and the expansion ansatz

The stationary states satisfy

$$\hat{H}_0\psi_i(\mathbf{r}) = E_i\psi_i(\mathbf{r}), \quad i \in \{1, 2\}. \quad (18)$$

Hence a general state can be expanded as

$$\Psi(\mathbf{r}, t) = c_1(t)\psi_1(\mathbf{r})e^{-iE_1t/\hbar} + c_2(t)\psi_2(\mathbf{r})e^{-iE_2t/\hbar}. \quad (19)$$

Info: Meaning of $c_1(t)$, $c_2(t)$

$|c_1(t)|^2$ and $|c_2(t)|^2$ are the instantaneous probabilities of being in $|1\rangle$ and $|2\rangle$ *provided the state is a coherent superposition*. (For statistical mixtures one often uses density matrices.)

4.3 Derive the coupled amplitude equations step-by-step

Start from (14) with (17) and (19).

4.3.1 Substitute the expansion into the TDSE

Compute the RHS:

$$i\hbar\frac{\partial\Psi}{\partial t} = i\hbar\left[\dot{c}_1\psi_1e^{-iE_1t/\hbar} + c_1\psi_1\left(-\frac{iE_1}{\hbar}\right)e^{-iE_1t/\hbar}\right] \quad (20)$$

$$+ i\hbar\left[\dot{c}_2\psi_2e^{-iE_2t/\hbar} + c_2\psi_2\left(-\frac{iE_2}{\hbar}\right)e^{-iE_2t/\hbar}\right] \quad (21)$$

$$= i\hbar\dot{c}_1\psi_1e^{-iE_1t/\hbar} + E_1c_1\psi_1e^{-iE_1t/\hbar} + i\hbar\dot{c}_2\psi_2e^{-iE_2t/\hbar} + E_2c_2\psi_2e^{-iE_2t/\hbar}. \quad (22)$$

Compute the LHS:

$$\hat{H}\Psi = (\hat{H}_0 + \hat{V})\left(c_1\psi_1e^{-iE_1t/\hbar} + c_2\psi_2e^{-iE_2t/\hbar}\right) \quad (23)$$

$$= c_1(\hat{H}_0\psi_1)e^{-iE_1t/\hbar} + c_2(\hat{H}_0\psi_2)e^{-iE_2t/\hbar} + c_1\hat{V}\psi_1e^{-iE_1t/\hbar} + c_2\hat{V}\psi_2e^{-iE_2t/\hbar} \quad (24)$$

$$= c_1E_1\psi_1e^{-iE_1t/\hbar} + c_2E_2\psi_2e^{-iE_2t/\hbar} + c_1\hat{V}\psi_1e^{-iE_1t/\hbar} + c_2\hat{V}\psi_2e^{-iE_2t/\hbar}, \quad (25)$$

where we used $\hat{H}_0\psi_i = E_i\psi_i$.

4.3.2 Cancel identical terms

Equate (25) and (22). The terms $c_1E_1\psi_1e^{-iE_1t/\hbar}$ and $c_2E_2\psi_2e^{-iE_2t/\hbar}$ appear on both sides and cancel, leaving

$$c_1\hat{V}\psi_1e^{-iE_1t/\hbar} + c_2\hat{V}\psi_2e^{-iE_2t/\hbar} = i\hbar\dot{c}_1\psi_1e^{-iE_1t/\hbar} + i\hbar\dot{c}_2\psi_2e^{-iE_2t/\hbar}. \quad (26)$$

4.3.3 Project onto each basis state

Multiply (26) by ψ_1^* and integrate over space. Use orthonormality

$$\int \psi_i^*(\mathbf{r})\psi_j(\mathbf{r}) d^3r = \delta_{ij}.$$

Define matrix elements

$$V_{ij}(t) \equiv \langle i|\hat{V}(t)|j\rangle = \int \psi_i^*\hat{V}(t)\psi_j d^3r. \quad (27)$$

Then we get

$$\dot{c}_1(t) = -\frac{i}{\hbar} [c_1(t)V_{11}(t) + c_2(t)V_{12}(t)e^{-i\omega_0 t}], \quad (28)$$

$$\dot{c}_2(t) = -\frac{i}{\hbar} [c_1(t)V_{21}(t)e^{+i\omega_0 t} + c_2(t)V_{22}(t)]. \quad (29)$$

Key point: What we achieved

We reduced the TDSE to **two coupled first-order ODEs** for the probability amplitudes.

5 Semi-classical light–matter interaction: dipole coupling

5.1 Dipole approximation

In the semi-classical approach, the field is classical and the atom is quantum. The perturbation is the dipole energy shift:

$$\hat{V}(t) = e\mathbf{r} \cdot \mathbf{E}(t). \quad (30)$$

(Here e is the magnitude of electron charge; the atomic dipole operator is $\hat{\boldsymbol{\mu}} = -e\mathbf{r}$.)

Assume linearly polarized monochromatic field:

$$\mathbf{E}(t) = \hat{\mathbf{e}} E_0 \cos(\omega t) = \hat{\mathbf{e}} \frac{E_0}{2} (e^{i\omega t} + e^{-i\omega t}). \quad (31)$$

5.2 Matrix elements and selection structure

Define the dipole matrix element

$$\boldsymbol{\mu}_{12} \equiv \langle 1|(-e\mathbf{r})|2\rangle, \quad \mu_{12} \equiv \boldsymbol{\mu}_{12} \cdot \hat{\mathbf{e}}. \quad (32)$$

Then

$$V_{12}(t) = \langle 1|e\mathbf{r} \cdot \mathbf{E}(t)|2\rangle = -\langle 1|\hat{\boldsymbol{\mu}} \cdot \mathbf{E}(t)|2\rangle = -\mu_{12} E_0 \cos(\omega t), \quad (33)$$

and similarly $V_{21}(t) = V_{12}^*(t)$ for real dipoles.

Info: Often-used simplifications

For many allowed electric-dipole transitions:

$$V_{11}(t) \approx V_{22}(t) \approx 0,$$

i.e. the diagonal Stark shifts are neglected at first pass. When needed, they reappear as AC Stark shifts and can explain spectral features (e.g. dressed-state splitting).

With $V_{11} = V_{22} = 0$, (28)–(29) simplify:

$$\dot{c}_1(t) = -\frac{i}{\hbar} c_2(t) V_{12}(t) e^{-i\omega_0 t}, \quad (34)$$

$$\dot{c}_2(t) = -\frac{i}{\hbar} c_1(t) V_{21}(t) e^{+i\omega_0 t}. \quad (35)$$

6 Rotating-wave approximation (RWA) and the Rabi frequency

Insert $V_{12}(t) \propto \cos(\omega t)$. Because $\cos(\omega t)$ contains $e^{\pm i\omega t}$, each equation contains terms oscillating at sums and differences of frequencies.

Info: Idea of RWA

Near resonance ($\omega \approx \omega_0$), the slowly varying terms oscillate at $|\omega - \omega_0|$, whereas the fast terms oscillate at $\omega + \omega_0 \approx 2\omega_0$ and average to (almost) zero over relevant time scales.

Define the (on-resonance) Rabi frequency:

$$\Omega_R \equiv \frac{|\mu_{12}| E_0}{\hbar}. \quad (36)$$

(Depending on convention, factors of 1/2 appear when you carry $\cos(\omega t) = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t})$ explicitly; we will keep the conventional RWA form below.)

Under RWA, the coupled equations become (exact resonance shown first):

$$\dot{c}_1(t) = i \frac{\Omega_R}{2} c_2(t), \quad (37)$$

$$\dot{c}_2(t) = i \frac{\Omega_R}{2} c_1(t), \quad (38)$$

and with detuning $\delta\omega = \omega - \omega_0$:

$$\dot{c}_1(t) = i \frac{\Omega_R}{2} e^{i\delta\omega t} c_2(t), \quad (39)$$

$$\dot{c}_2(t) = i \frac{\Omega_R}{2} e^{-i\delta\omega t} c_1(t). \quad (40)$$

Key point: Interpretation of Ω_R

Ω_R is a **coherent oscillation rate**: how fast population swaps between $|1\rangle$ and $|2\rangle$ when driven resonantly.

7 Weak-field limit: recover Einstein's B coefficient from the TDSE picture

7.1 Physical regime

Weak-field means the atom is only slightly perturbed during the interaction time t :

$$|c_2(t)|^2 \ll 1, \quad c_1(t) \approx 1,$$

so excitation can be computed to first order (time-dependent perturbation theory).

7.2 First-order solution (step-by-step)

Start with

$$\dot{c}_2(t) \approx -\frac{i}{\hbar} V_{21}(t) e^{i\omega_0 t},$$

using $c_1 \approx 1$ in the exact (non-RWA) equation.

Integrate from 0 to t :

$$c_2(t) - c_2(0) = -\frac{i}{\hbar} \int_0^t V_{21}(t') e^{i\omega_0 t'} dt'. \quad (41)$$

Assume initial ground state $c_2(0) = 0$. Use $V_{21}(t) = -\mu_{21} E_0 \cos(\omega t)$:

$$c_2(t) = \frac{i\mu_{21} E_0}{\hbar} \int_0^t \cos(\omega t') e^{i\omega_0 t'} dt' \quad (42)$$

$$= \frac{i\mu_{21} E_0}{2\hbar} \int_0^t (e^{i\omega t'} + e^{-i\omega t'}) e^{i\omega_0 t'} dt' \quad (43)$$

$$= \frac{i\mu_{21} E_0}{2\hbar} \int_0^t (e^{i(\omega_0 + \omega)t'} + e^{i(\omega_0 - \omega)t'}) dt' \quad (44)$$

$$= \frac{i\mu_{21} E_0}{2\hbar} \left[\frac{e^{i(\omega_0 + \omega)t} - 1}{i(\omega_0 + \omega)} + \frac{e^{i(\omega_0 - \omega)t} - 1}{i(\omega_0 - \omega)} \right] \quad (45)$$

$$= \frac{\mu_{21} E_0}{2\hbar} \left[\frac{e^{i(\omega_0 + \omega)t} - 1}{\omega_0 + \omega} + \frac{e^{i(\omega_0 - \omega)t} - 1}{\omega_0 - \omega} \right]. \quad (46)$$

Near resonance, $|\omega_0 - \omega| \ll \omega_0 + \omega$, so the second term dominates. Define $\delta\omega = \omega - \omega_0$ so $\omega_0 - \omega = -\delta\omega$:

$$c_2(t) \approx \frac{\mu_{21} E_0}{2\hbar} \cdot \frac{e^{-i\delta\omega t} - 1}{-\delta\omega} = \frac{\mu_{21} E_0}{2\hbar} \cdot \frac{1 - e^{-i\delta\omega t}}{\delta\omega}. \quad (47)$$

Compute transition probability:

$$P_{1 \rightarrow 2}(t) = |c_2(t)|^2 \approx \left(\frac{|\mu_{12}| E_0}{2\hbar} \right)^2 \left| \frac{1 - e^{-i\delta\omega t}}{\delta\omega} \right|^2. \quad (48)$$

Use identity:

$$|1 - e^{-ix}|^2 = (1 - e^{-ix})(1 - e^{ix}) = 2 - 2\cos x = 4\sin^2(x/2).$$

Thus

$$P_{1 \rightarrow 2}(t) \approx \left(\frac{|\mu_{12}| E_0}{\hbar} \right)^2 \frac{\sin^2(\delta\omega t/2)}{\delta\omega^2}. \quad (49)$$

Key point: Weak-field scaling

From (49), $P_{1 \rightarrow 2}(t) \propto |\mu_{12}|^2 E_0^2$ (dipole squared & field intensity). This is exactly the scaling that the Einstein B coefficient encodes.

7.3 Connecting to Einstein B

Einstein absorption rate (per atom) is proportional to $B_{12}^\omega u(\omega)$. In the semiclassical wave picture, $u(\omega)$ is proportional to field intensity and hence E_0^2 . Therefore, matching the proportionalities connects B to $|\mu_{12}|^2$ (dipole matrix elements).

Logic note (for lecture): Teaching bridge

Say: “Einstein B is the *incoherent* rate language for what, microscopically, is the *coherent* dipole coupling strength in the TDSE.”

8 Strong-field limit: Rabi oscillations

8.1 Solve the resonant RWA equations explicitly

Assume exact resonance $\delta\omega = 0$ and use RWA:

$$\dot{c}_1 = i\frac{\Omega_R}{2}c_2, \quad (50)$$

$$\dot{c}_2 = i\frac{\Omega_R}{2}c_1. \quad (51)$$

Differentiate the first equation:

$$\ddot{c}_1 = i\frac{\Omega_R}{2}\dot{c}_2 = i\frac{\Omega_R}{2}\left(i\frac{\Omega_R}{2}c_1\right) = -\left(\frac{\Omega_R}{2}\right)^2 c_1. \quad (52)$$

So c_1 satisfies the harmonic equation:

$$\ddot{c}_1 + \left(\frac{\Omega_R}{2}\right)^2 c_1 = 0, \quad (53)$$

with solution

$$c_1(t) = A \cos(\Omega_R t/2) + B \sin(\Omega_R t/2). \quad (54)$$

Similarly, from $\dot{c}_1 = i(\Omega_R/2)c_2$,

$$c_2(t) = \frac{2}{i\Omega_R}\dot{c}_1(t). \quad (55)$$

For initial ground state: $c_1(0) = 1$, $c_2(0) = 0$. Then $c_1(0) = A = 1$. Also $c_2(0) = \frac{2}{i\Omega_R}\dot{c}_1(0) = 0 \Rightarrow \dot{c}_1(0) = 0 \Rightarrow B = 0$.

Hence

$$c_1(t) = \cos(\Omega_R t/2), \quad (56)$$

$$c_2(t) = i \sin(\Omega_R t/2). \quad (57)$$

Therefore populations oscillate:

$$P_1(t) = |c_1(t)|^2 = \cos^2(\Omega_R t/2), \quad (58)$$

$$P_2(t) = |c_2(t)|^2 = \sin^2(\Omega_R t/2). \quad (59)$$

Key point: Rabi flopping

A resonant, strong coherent field drives **periodic population transfer** between $|1\rangle$ and $|2\rangle$ at angular frequency Ω_R .

8.2 Pulse area language

Define pulse area

$$\Theta \equiv \int \Omega_R(t) dt. \quad (60)$$

A π -pulse inverts the system ($P_2 = 1$). A 2π pulse returns it to ground ($P_2 = 0$).

Logic note (for lecture): Easy experimental narrative

Explain why fluorescence is strong after a π pulse (system ends excited and then decays), and weak after 2π (system returns to ground).

9 Damping, dephasing, and the crossover back to Einstein behaviour

Real systems are not perfectly coherent:

- **Population relaxation** (energy decay) with characteristic T_1 .
- **Dephasing** (loss of phase coherence) with characteristic T_2 .

A simple phenomenological inclusion is to add a damping rate γ to the coherence dynamics, producing **damped Rabi oscillations**.

Info: When do we actually see oscillations?

Rabi oscillations are observable when the coherent drive beats damping:

$$\Omega_R \gg \gamma.$$

If γ is large (weak field, strong damping), oscillations are washed out and the behaviour looks like *incoherent transitions* (Einstein picture).

9.1 Asymptotic weak-field consistency check

In the strongly damped regime, the excited population tends to a small steady value proportional to Ω_R^2/γ^2 :

$$|c_2|^2 \rightarrow \frac{\Omega_R^2}{4\gamma^2} = \frac{\mu_{12}^2 E_0^2}{4\hbar^2 \gamma^2}, \quad (61)$$

consistent with the weak-field scaling $|c_2|^2 \propto \mu_{12}^2 E_0^2$ and with Einstein's rate picture once one identifies γ with radiative decay/broadening scales.

Logic note (for lecture): Big message

“Einstein coefficients become valid when coherence is quickly destroyed. Coherent TDSE physics becomes visible when Ω_R outruns γ .”

10 Experimental observations and the Mollow triplet**10.1 Direct observations of Rabi oscillations**

Experiments measure oscillatory signatures versus pulse area Θ (e.g. fluorescence or photocurrent), showing maxima at odd multiples of π and minima at even multiples.

10.2 Mollow triplet: frequency-domain signature of strong driving

In the strong-field regime, the emission spectrum can split into three peaks:

$$\omega_0, \quad \omega_0 \pm \Omega_R.$$

This is the **Mollow triplet** and is the frequency-domain counterpart of time-domain Rabi oscillations.

Info: Dressed-state interpretation (high level)

An intense resonant field mixes the bare atomic states into *dressed states* separated by $\hbar\Omega_R$ (AC Stark splitting). Radiative transitions among dressed states yield the three spectral lines.

Key point: Why this matters for “quantum tech”

Spectral splitting directly reveals the coherent coupling strength Ω_R and is a diagnostic for entering the strong-driving regime.

11 Part III: Bloch sphere representation of a two-level state**11.1 General state and normalization**

A general two-level pure state:

$$|\psi\rangle = c_1 |1\rangle + c_2 |2\rangle, \quad |c_1|^2 + |c_2|^2 = 1. \quad (62)$$

Parameterize amplitudes using two angles:

$$c_1 = \cos \frac{\theta}{2}, \quad c_2 = e^{i\phi} \sin \frac{\theta}{2}, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi. \quad (63)$$

11.2 Bloch vector

Define the Bloch vector $\mathbf{s} = (s_x, s_y, s_z)$ (expectation of Pauli operators):

$$s_i = \langle \sigma_i \rangle, \quad i \in \{x, y, z\}. \quad (64)$$

Using (63), one obtains

$$s_x = \sin \theta \cos \phi, \quad (65)$$

$$s_y = \sin \theta \sin \phi, \quad (66)$$

$$s_z = \cos \theta, \quad (67)$$

so \mathbf{s} lies on the unit sphere.

Key point: Geometric meaning

$|1\rangle \leftrightarrow$ south/north pole depending on convention, $|2\rangle \leftrightarrow$ opposite pole, superpositions \leftrightarrow points on the Bloch sphere. Rotations on the Bloch sphere correspond to coherent control (e.g. Rabi pulses).

11.3 Driven two-level Hamiltonian as a rotation generator (conceptual)

In a suitable rotating frame and under RWA, the driven two-level Hamiltonian takes the standard form

$$\hat{H}_{\text{rot}} = \frac{\hbar}{2} (\delta\omega \sigma_z + \Omega_R \sigma_x) \quad (\text{choice of axis depends on phase convention}). \quad (68)$$

Thus the state vector undergoes precession about an effective field vector

$$\mathbf{\Omega}_{\text{eff}} = (\Omega_R, 0, \delta\omega),$$

giving a direct geometric picture for:

- **On resonance** ($\delta\omega = 0$): rotation about x -axis at rate Ω_R .
- **Detuned** ($\delta\omega \neq 0$): rotation about a tilted axis.
- **Damping/dephasing**: shrinking of Bloch vector toward equilibrium (Bloch equations).

Logic note (for lecture): How to connect to Einstein coefficients in one sentence

“In the Bloch picture, strong damping collapses the transverse components (T_2 short), destroying coherence; the remaining dynamics reduces to population rate equations governed by A and B .”

Mini-summary sheet (for the last slide)

- Einstein picture: **rates** for absorption ($B_{12}u$), stimulated emission ($B_{21}u$), spontaneous emission (A_{21}).
- Detailed balance + Planck spectrum \Rightarrow relations among A and B :

$$g_1 B_{12}^\omega = g_2 B_{21}^\omega, \quad A_{21} = \frac{\hbar\omega^3}{\pi^2 c^3} B_{21}^\omega.$$

- TDSE picture: **amplitudes** $c_1(t), c_2(t)$ obey coupled ODEs derived by projection.
- Dipole coupling: $\hat{V}(t) = e\mathbf{r} \cdot \mathbf{E}(t)$; coherent strength set by μ_{12} .

- Weak-field: $P_{1 \rightarrow 2} \propto |\mu_{12}|^2 E_0^2 \Rightarrow$ connects to B .
- Strong-field: **Rabi oscillations** $P_2(t) = \sin^2(\Omega_R t/2)$.
- Damping: oscillations observable when $\Omega_R \gg \gamma$.
- Strong-field spectrum: **Mollow triplet** at ω_0 and $\omega_0 \pm \Omega_R$.
- Bloch sphere: $|\psi\rangle = \cos(\theta/2) |1\rangle + e^{i\phi} \sin(\theta/2) |2\rangle$; drives generate rotations.

Attribution: The core structure and several key equations follow Fox, *Quantum Optics: An Introduction* (Oxford, 2006), especially Ch. 4 (Einstein coefficients) and Ch. 9 (resonant light–atom interactions).