Photonic Integrated Circuits
Lecture 1: Maxwell Equations, Wave Equation, Spectral Representation, Monochromatic Fields

Contents

1 Maxwell Equations in Differential and Integral Form

1.1 Notation

Let $\mathbf{r} = (x, y, z)$ and t denote space and time. Fields: $\mathbf{E}(\mathbf{r}, t)$ (electric field), $\mathbf{H}(\mathbf{r}, t)$ (magnetic field intensity), $\mathbf{D}(\mathbf{r}, t)$ (electric displacement), $\mathbf{B}(\mathbf{r}, t)$ (magnetic flux density), $\rho(\mathbf{r}, t)$ (free charge density), $\mathbf{j}(\mathbf{r}, t)$ (free current density).

Differential operators:

$$\nabla \equiv \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix}, \tag{1}$$

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}, \tag{2}$$

$$\nabla \times \mathbf{F} = \begin{bmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{bmatrix}, \tag{3}$$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}. \tag{4}$$

We will use the vector identity:

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}. \tag{5}$$

1.2 Integral theorems (written explicitly)

Gauss (divergence) theorem

Let V be a volume with closed boundary surface ∂V and outward unit normal $\mathbf{n}$. For a sufficiently smooth vector field $\mathbf{F}(\mathbf{r})$,

$$\int_{\partial V} \mathbf{F}(\mathbf{r}) \cdot \mathbf{n} \; \mathrm{d}a = \int_V (\nabla \cdot \mathbf{F}(\mathbf{r})) \; \mathrm{d}V. \tag{6}$$

Stokes theorem

Let A be an oriented surface with boundary curve ∂A and unit normal $\mathbf{n}$ consistent with the right-hand rule. For a sufficiently smooth vector field $\mathbf{F}(\mathbf{r})$,

$$\oint_{\partial A} \mathbf{F}(\mathbf{r}) \cdot \; \mathrm{d}\mathbf{s} = \int_A (\nabla \times \mathbf{F}(\mathbf{r})) \cdot \mathbf{n} \; \mathrm{d}a. \tag{7}$$

1.3 Maxwell equations in integral form

(I) Gauss law for electric displacement

$$\int_{\partial V} \mathbf{D}(\mathbf{r}, t) \cdot \mathbf{n} \; \mathrm{d}a = \int_V \rho(\mathbf{r}, t) \; \mathrm{d}V. \tag{8}$$

(II) Faraday law

$$\oint_{\partial A} \mathbf{E}(\mathbf{r}, t) \cdot \; \mathrm{d}\mathbf{s} = -\frac{\partial}{\partial t} \int_A \mathbf{B}(\mathbf{r}, t) \cdot \mathbf{n} \; \mathrm{d}a. \tag{9}$$

(III) Ampère–Maxwell law

$$\oint_{\partial A} \mathbf{H}(\mathbf{r},t) \cdot \, \mathrm{d}\mathbf{s} = \int_A \left[\mathbf{j}(\mathbf{r},t) + \frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t} \right] \cdot \mathbf{n} \, \mathrm{d}a. \tag{10}$$

(IV) Gauss law for magnetism

$$\int_{\partial V} \mathbf{B}(\mathbf{r},t) \cdot \mathbf{n} \, \mathrm{d}a = 0. \tag{11}$$

1.4 Derive differential Maxwell equations (all steps)

1.4.1 From (8) to $\nabla \cdot \mathbf{D} = \rho$

Start from (8):

$$\int_{\partial V} \mathbf{D}(\mathbf{r},t) \cdot \mathbf{n} \, \mathrm{d}a = \int_V \rho(\mathbf{r},t) \, \mathrm{d}V. \tag{12}$$

Apply Gauss theorem (6) to the left-hand side:

$$\int_V \left(\nabla \cdot \mathbf{D}(\mathbf{r},t) \right) \, \mathrm{d}V = \int_V \rho(\mathbf{r},t) \, \mathrm{d}V. \tag{13}$$

Bring to one side:

$$\int_V \left(\nabla \cdot \mathbf{D}(\mathbf{r},t) - \rho(\mathbf{r},t) \right) \, \mathrm{d}V = 0. \tag{14}$$

Since V is arbitrary, the integrand must vanish pointwise:

$$\nabla \cdot \mathbf{D}(\mathbf{r},t) = \rho(\mathbf{r},t). \tag{15}$$

1.4.2 From (9) to $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$

Start from (9):

$$\oint_{\partial A} \mathbf{E} \cdot \, \mathrm{d}\mathbf{s} = -\frac{\partial}{\partial t} \int_A \mathbf{B} \cdot \mathbf{n} \, \mathrm{d}a. \tag{16}$$

Apply Stokes theorem (7) to the left-hand side:

$$\int_A (\nabla \times \mathbf{E}) \cdot \mathbf{n} \, \mathrm{d}a = -\frac{\partial}{\partial t} \int_A \mathbf{B} \cdot \mathbf{n} \, \mathrm{d}a. \tag{17}$$

Move the time derivative inside the surface integral:

$$-\frac{\partial}{\partial t} \int_A \mathbf{B} \cdot \mathbf{n} \, \mathrm{d}a = -\int_A \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \, \mathrm{d}a. \tag{18}$$

Therefore:

$$\int_A \left[(\nabla \times \mathbf{E}) + \frac{\partial \mathbf{B}}{\partial t} \right] \cdot \mathbf{n} \, \mathrm{d}a = 0. \tag{19}$$

Since A is arbitrary:

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}. \tag{20}$$

1.4.3 From (10) to $\nabla \times \mathbf{H} = \mathbf{j} + \partial \mathbf{D}/\partial t$

Start from (10):

$$\oint_{\partial A} \mathbf{H} \cdot \, \mathrm{d}\mathbf{s} = \int_A \left(\mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot \mathbf{n} \, \mathrm{d}a. \tag{21}$$

Apply Stokes theorem (7):

$$\int_A (\nabla \times \mathbf{H}) \cdot \mathbf{n} \, \mathrm{d}a = \int_A \left(\mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot \mathbf{n} \, \mathrm{d}a. \tag{22}$$

Bring to one side:

$$\int_A \left[(\nabla \times \mathbf{H}) - \mathbf{j} - \frac{\partial \mathbf{D}}{\partial t} \right] \cdot \mathbf{n} \, \mathrm{d}a = 0. \tag{23}$$

Since A is arbitrary:

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \mathbf{j}(\mathbf{r}, t) + \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t}. \tag{24}$$

1.4.4 From (11) to $\nabla \cdot \mathbf{B} = 0$

Start from (11):

$$\int_{\partial V} \mathbf{B} \cdot \mathbf{n} \, \mathrm{d}a = 0. \tag{25}$$

Apply Gauss theorem (6):

$$\int_V (\nabla \cdot \mathbf{B}) \, \mathrm{d}V = 0. \tag{26}$$

Since V is arbitrary:

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0. \tag{27}$$

1.5 Maxwell equations (differential form) collected

$$\boxed{\begin{aligned} \nabla \cdot \mathbf{D} &= \rho, \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \times \mathbf{H} &= \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}, \\ \nabla \cdot \mathbf{B} &= 0. \end{aligned}} \tag{28}$$

1.6 Constitutive relations

In macroscopic media:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \tag{29}$$

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}). \tag{30}$$

In linear isotropic media:

$$\mathbf{D} = \varepsilon \mathbf{E}, \qquad \mathbf{B} = \mu \mathbf{H}. \tag{31}$$

2 Maxwell Equations for a Medium Without Charge and Current

Assume a source-free region:

$$\rho(\mathbf{r},t) = 0, \qquad \mathbf{j}(\mathbf{r},t) = 0. \tag{32}$$

Then:

$$\boxed{\begin{aligned} \nabla \cdot \mathbf{D} &= 0, \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t}, \\ \nabla \cdot \mathbf{B} &= 0. \end{aligned}} \tag{33}$$

If ε, μ are constants:

$$\nabla \cdot (\varepsilon \mathbf{E}) = 0 \Rightarrow \varepsilon(\nabla \cdot \mathbf{E}) = 0 \Rightarrow \nabla \cdot \mathbf{E} = 0, \tag{34}$$

$$\nabla \cdot (\mu \mathbf{H}) = 0 \Rightarrow \mu(\nabla \cdot \mathbf{H}) = 0 \Rightarrow \nabla \cdot \mathbf{H} = 0. \tag{35}$$

3 Wave Equation and Helmholtz Equation

3.1 Wave equation for E (curl–curl form)

Start from Faraday (source-free):

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{36}$$

With $\mathbf{B} = \mu \mathbf{H}$:

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}. \tag{37}$$

Take curl:

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu \frac{\partial}{\partial t}(\nabla \times \mathbf{H}). \tag{38}$$

Use Ampère–Maxwell (source-free):

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}. \tag{39}$$

With $\mathbf{D} = \varepsilon \mathbf{E}$:

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}. \tag{40}$$

Substitute:

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{41}$$

Bring to one side:

$$\nabla \times (\nabla \times \mathbf{E}) + \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0. \tag{42}$$

3.2 Wave equation for E (Laplacian form)

Use identity (5):

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}. \tag{43}$$

In homogeneous source-free region, $\nabla \cdot \mathbf{E} = 0$:

$$-\nabla^2 \mathbf{E} + \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \;\Rightarrow\; \nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0. \tag{44}$$

Let $v = 1/\sqrt{\mu\varepsilon}$:

$$\nabla^2 \mathbf{E} - \frac{1}{v^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0. \tag{45}$$

3.3 Helmholtz equation (time-harmonic reduction)

Assume $\tilde{\mathbf{E}}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}$.

EM field oscillates sinusoidally in time at a single angular frequency ω everywhere, and that all spatial structure is time-independent.

1. The amplitude $|\mathbf{E}(\mathbf{r})|$ varies with position
2. The phase $\phi(\mathbf{r})$ varies with position
3. Continuous-wave (CW) regime.

$$\tilde{\mathbf{E}}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r})\, \mathrm{e}^{-i\omega t} \tag{46}$$

Then $\partial^2 \tilde{\mathbf{E}}/\partial t^2 = -\omega^2 \tilde{\mathbf{E}}$. Insert into (45):

$$\nabla^2 \mathbf{E} + \frac{\omega^2}{v^2} \mathbf{E} = 0. \tag{47}$$

Define $k = \omega/v = \omega\sqrt{\mu\varepsilon}$:

$$\nabla^2 \mathbf{E}(\mathbf{r}) + k^2 \mathbf{E}(\mathbf{r}) = 0. \tag{48}$$

4 Plane Waves, Evanescent Waves, and the General Homogeneous Solution

In this section we classify the fundamental solutions of the Helmholtz equation

$$\nabla^2 \mathbf{E}(\mathbf{r}) + k^2 \mathbf{E}(\mathbf{r}) = 0, \tag{49}$$

which governs monochromatic electromagnetic fields in homogeneous, source-free media. Understanding these solutions is essential because:

- Plane waves represent locally uniform propagation,
- Evanescent waves explain confinement and near-field behavior,
- Arbitrary fields can be constructed as superpositions of these elementary solutions.

4.1 Plane-wave ansatz and dispersion relation

Motivation. In a homogeneous medium, the material parameters ε and μ are spatially constant. Therefore, the governing equation (49) is *translation invariant*: if $\mathbf{E}(\mathbf{r})$ is a solution, so is $\mathbf{E}(\mathbf{r}+\mathbf{r}_0)$. The natural eigenfunctions of a translation-invariant linear differential operator are complex exponentials. This motivates the plane-wave ansatz

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}, \tag{50}$$

where:

- $\mathbf{E}_0$ is a constant (possibly complex) vector amplitude,
- $\mathbf{k}$ is the wave vector, determining propagation direction and wavelength.

Action of the Laplacian. Applying the gradient operator to the exponential,

$$\nabla \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} = \mathrm{i}\mathbf{k}\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}},$$

and applying it twice gives

$$\nabla^2 \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} = -\,(\mathbf{k}\cdot\mathbf{k})\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}.$$

Since $\mathbf{E}_0$ is constant in space,

$$\nabla^2 \mathbf{E}(\mathbf{r}) = -\,(\mathbf{k}\cdot\mathbf{k})\, \mathbf{E}(\mathbf{r}). \tag{51}$$

Dispersion relation. Substituting into the Helmholtz equation (49),

$$-\,(\mathbf{k}\cdot\mathbf{k})\, \mathbf{E} + k^2 \mathbf{E} = 0. \tag{52}$$

For a nontrivial field $\mathbf{E} \neq 0$, the coefficient must vanish, yielding

$$\boxed{\mathbf{k}\cdot\mathbf{k} = k^2.} \tag{53}$$

Physical meaning. Equation (53) is the *dispersion relation* of a homogeneous, isotropic medium. It states that:

- the magnitude of the spatial oscillation vector $\mathbf{k}$ is fixed by the frequency,
- $k = \omega\sqrt{\mu\varepsilon}$ links temporal oscillation to spatial variation,
- the wave propagates without distortion in the absence of boundaries or inhomogeneities.

4.2 Transversality condition

Starting point. In a homogeneous, source-free region,

$$\nabla\cdot\mathbf{E} = 0. \tag{54}$$

This expresses the absence of free charge and is one of Maxwell's equations.

Apply to a plane wave. Using the plane-wave form (50),

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \nabla \cdot \left(\mathbf{E}_0 \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\right) \\ &= \mathbf{E}_0 \cdot \nabla \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \\ &= \mathrm{i}(\mathbf{k}\cdot\mathbf{E}_0)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}. \end{aligned} \tag{55}$$

For this to vanish everywhere in space, we must have

$$\boxed{\mathbf{k}\cdot\mathbf{E}_0 = 0.} \tag{56}$$

Physical interpretation. Equation (56) means that:

- the electric field oscillates *perpendicular* to the direction of propagation,
- electromagnetic plane waves in homogeneous media are *transverse waves*,
- there is no longitudinal electric-field component in free space or uniform dielectrics.

This result directly follows from charge conservation and has no analogue in scalar wave theory.

4.3 Evanescent waves

Wave-vector decomposition. Write the wave vector as

$$\mathbf{k} = (k_x, k_y, k_z), \tag{57}$$

so that the dispersion relation (53) becomes

$$k^2 = k_x^2 + k_y^2 + k_z^2. \tag{58}$$

Propagation versus decay. Solving for the longitudinal component,

$$k_z^2 = k^2 - (k_x^2 + k_y^2). \tag{59}$$

Two physically distinct cases arise:

- If $k_x^2 + k_y^2 < k^2$, then k_z is real and the wave propagates in z.
- If $k_x^2 + k_y^2 > k^2$, then k_z is purely imaginary.

Evanescent regime. In the latter case, define

$$k_z = \mathrm{i}\alpha, \qquad \alpha = \sqrt{(k_x^2 + k_y^2) - k^2} > 0. \tag{60}$$

The plane-wave factor becomes

$$\mathrm{e}^{\mathrm{i}k_z z} = \mathrm{e}^{-\alpha z},$$

so the field takes the form

$$\boxed{\mathbf{E}(\mathbf{r}) = \mathbf{E}_0\, \mathrm{e}^{\mathrm{i}(k_x x + k_y y)}\, \mathrm{e}^{-\alpha z}.} \tag{61}$$

Physical interpretation. Evanescent waves:

- do not transport energy in the decay direction,
- arise near interfaces, waveguides, and total internal reflection,
- are responsible for optical confinement and near-field effects,
- are essential for understanding guided modes and tunneling.

4.4 General homogeneous solution (superposition principle)

Linearity and completeness. The Helmholtz equation is linear, so any linear combination of solutions is also a solution. Moreover, plane waves form a complete basis for solutions in homogeneous space.

Discrete superposition. A field composed of a discrete set of modes can be written as

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left\{\sum_n \mathbf{E}_n\, \mathrm{e}^{\mathrm{i}\mathbf{k}_n\cdot\mathbf{r}}\mathrm{e}^{-\mathrm{i}\omega_n t}\right\}. \tag{62}$$

This form is typical for cavity modes and resonant systems.

Continuous superposition (Fourier representation). More generally, an arbitrary homogeneous-field solution can be expressed as a continuous superposition:

$$\boxed{\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left\{\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \mathbf{E}_0(\mathbf{k},\omega)\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\mathrm{e}^{-\mathrm{i}\omega t}\ \mathrm{d}^3k\ \mathrm{d}\omega\right\}.} \tag{63}$$

Connection to later methods. This representation provides the conceptual bridge to:

- modal expansions in waveguides,
- spectral-domain solvers,
- time-domain methods such as FDTD (where each frequency component evolves simultaneously).

5 Spectral (Frequency) Representation of Maxwell Equations

In this section we reformulate Maxwell's equations in the frequency domain. This representation is fundamental for:

- monochromatic (steady-state) fields,
- modal and eigenvalue solvers,
- frequency-domain scattering problems,
- understanding dispersion and material response.

It also provides the conceptual bridge between time-domain solvers (FDTD) and frequency-domain solvers (FDE, FEM, EME).

5.1 Why move to the frequency domain? (physical motivation)

Maxwell's equations in the time domain describe *dynamics*: how fields evolve in time given sources and material properties. However, many electromagnetic problems of interest are:

- driven by sources at well-defined frequencies,
- observed after all transients have died out,
- linear and time-invariant.

In such cases, it is natural to decompose the fields into their frequency components. Physically, this means:

> *Instead of following the full time evolution, we analyze how the system responds to each frequency independently.*

This is possible because Maxwell's equations are linear.

5.2 Fourier transform pair (definition and interpretation)

Definition. We define the temporal Fourier transform of the electric field as

$$\hat{\mathbf{E}}(\mathbf{r},\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r},t)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}t, \tag{64}$$

with the inverse transform

$$\mathbf{E}(\mathbf{r},t) = \int_{-\infty}^{\infty} \hat{\mathbf{E}}(\mathbf{r},\omega)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}\omega. \tag{65}$$

Analogous definitions apply to $\mathbf{H}, \mathbf{D}, \mathbf{B}, \rho$, and $\mathbf{j}$.

Physical meaning. These relations state that:

- $\hat{\mathbf{E}}(\mathbf{r},\omega)$ is the *complex amplitude* of the field oscillating at angular frequency ω,
- $\mathbf{E}(\mathbf{r},t)$ is the superposition of all frequency components,
- time localization (pulses) corresponds to broadband spectra,
- pure sinusoidal oscillation corresponds to a delta function in frequency.

Thus, the frequency domain provides a *spectral decomposition* of the electromagnetic field.

5.3 Action of time derivatives under Fourier transform

Key mathematical property. For a sufficiently well-behaved function $f(t)$,

$$\mathcal{F}\left\{\frac{\partial f}{\partial t}\right\} = -\mathrm{i}\omega\, \hat{f}(\omega). \tag{66}$$

Physical interpretation. This means:

- time differentiation becomes simple multiplication in frequency,
- temporal dynamics are replaced by algebraic frequency factors,
- oscillations at higher frequency respond more strongly to time derivatives.

This property is the reason frequency-domain formulations are often simpler.

5.4 Derivation of frequency-domain Maxwell equations

Start from time-domain Maxwell equations. In macroscopic form:

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = \rho(\mathbf{r}, t), \tag{67}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0, \tag{68}$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}}{\partial t}, \tag{69}$$

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \mathbf{j}(\mathbf{r}, t) + \frac{\partial \mathbf{D}}{\partial t}. \tag{70}$$

Apply the Fourier transform in time. Since spatial derivatives act only on $\mathbf{r}$, they commute with the transform. Only time derivatives are affected.

- Divergence equations contain no time derivatives ⇒ unchanged.
- Curl equations contain first-order time derivatives ⇒ introduce factors of $-\mathrm{i}\omega$.

Applying (66):

$$\nabla \cdot \hat{\mathbf{D}}(\mathbf{r}, \omega) = \hat{\rho}(\mathbf{r}, \omega), \tag{71}$$

$$\nabla \cdot \hat{\mathbf{B}}(\mathbf{r}, \omega) = 0, \tag{72}$$

$$\nabla \times \hat{\mathbf{E}}(\mathbf{r}, \omega) = \mathrm{i}\omega\, \hat{\mathbf{B}}(\mathbf{r}, \omega), \tag{73}$$

$$\nabla \times \hat{\mathbf{H}}(\mathbf{r}, \omega) = \hat{\mathbf{j}}(\mathbf{r}, \omega) - \mathrm{i}\omega\, \hat{\mathbf{D}}(\mathbf{r}, \omega). \tag{74}$$

Collected frequency-domain Maxwell equations.

$$\boxed{\begin{aligned} \nabla \cdot \hat{\mathbf{D}}(\mathbf{r}, \omega) &= \hat{\rho}(\mathbf{r}, \omega), \\ \nabla \cdot \hat{\mathbf{B}}(\mathbf{r}, \omega) &= 0, \\ \nabla \times \hat{\mathbf{E}}(\mathbf{r}, \omega) &= \mathrm{i}\omega\hat{\mathbf{B}}(\mathbf{r}, \omega), \\ \nabla \times \hat{\mathbf{H}}(\mathbf{r}, \omega) &= \hat{\mathbf{j}}(\mathbf{r}, \omega) - \mathrm{i}\omega\hat{\mathbf{D}}(\mathbf{r}, \omega). \end{aligned}} \tag{75}$$

5.5 Constitutive relations and dispersion

In the frequency domain, linear material response becomes algebraic:

$$\hat{\mathbf{D}}(\mathbf{r}, \omega) = \varepsilon(\mathbf{r}, \omega)\hat{\mathbf{E}}(\mathbf{r}, \omega), \qquad \hat{\mathbf{B}}(\mathbf{r}, \omega) = \mu(\mathbf{r}, \omega)\hat{\mathbf{H}}(\mathbf{r}, \omega). \tag{76}$$

Important consequence. Material dispersion (frequency dependence of ε and μ) appears *naturally* in the frequency domain. In the time domain, the same physics would require temporal convolutions.

5.6 What is gained and what is lost?

Advantages.

- Time derivatives become algebraic.
- Steady-state problems reduce to boundary-value or eigenvalue problems.
- Modal analysis and resonances are naturally described.
- Ideal for waveguides, cavities, scattering at fixed wavelength.

Limitations.

- Transient dynamics are removed.
- Pulses and switching cannot be described directly.
- Broadband behavior requires repeated solves at many frequencies.

5.7 Connection to later methods (important for PIC)

- **MODE / FDE**: solves Eq. (75) as an eigenvalue problem at fixed ω.
- **FDTD**: solves time-domain Maxwell equations, then Fourier-transforms fields to recover $\hat{\mathbf{E}}(\omega)$.
- **Helmholtz equation**: follows from Eq. (75) in homogeneous source-free regions.

Thus, the spectral representation is not a different theory, but a different *projection* of the same Maxwell equations.

6 Monochromatic Fields

In this section we specialize the spectral representation of electromagnetic fields to the important and widely used case of *monochromatic* excitation. This case underlies:

- waveguide mode analysis,
- resonator eigenmodes,
- steady-state scattering problems,
- frequency-domain solvers such as FDE and FEM.

Physically, we are describing electromagnetic fields that oscillate at a single angular frequency ω_0, with all transients having vanished.

6.1 Real physical field and the role of complex representation

Why complex fields are introduced. Maxwell's equations are linear and involve time derivatives. Representing sinusoidally oscillating fields using complex exponentials greatly simplifies algebra, because differentiation becomes multiplication by $-\mathrm{i}\omega$.

However, measurable electromagnetic fields must be *real-valued.* The complex representation is therefore a mathematical tool, not a physical observable.

Definition of the physical field. For a single frequency ω_0, we write the complex (phasor) field as

$$\mathbf{E}(\mathbf{r})\,\mathrm{e}^{-\mathrm{i}\omega_0 t},$$

and define the physical electric field as its real part:

$$\mathbf{E}_{\text{phys}}(\mathbf{r},t) = \operatorname{Re}\left\{\mathbf{E}(\mathbf{r})\,\mathrm{e}^{-\mathrm{i}\omega_0 t}\right\}. \tag{77}$$

Explicit real form. Using the identity $\operatorname{Re}\{A\} = (A + A^*)/2$, this can be written as

$$\boxed{\mathbf{E}_{\text{phys}}(\mathbf{r},t) = \frac{1}{2}\left[\mathbf{E}(\mathbf{r})\,\mathrm{e}^{-\mathrm{i}\omega_0 t} + \mathbf{E}^*(\mathbf{r})\,\mathrm{e}^{+\mathrm{i}\omega_0 t}\right].} \tag{78}$$

Physical interpretation. Equation (78) shows that:

- a real monochromatic field necessarily contains both $+\omega_0$ and $-\omega_0$ components,
- $\mathbf{E}(\mathbf{r})$ encodes the spatial amplitude and phase,
- the temporal behavior is purely harmonic and time-periodic,
- there is no energy growth or decay in time (steady state).

This is the electromagnetic analogue of a steady-state sinusoidal response in linear circuits.

6.2 Delta-function spectrum of a monochromatic field

Fourier transform of a single-frequency signal. A pure harmonic oscillation extends infinitely in time. As a consequence, it is infinitely narrow in frequency.

Mathematically, this is expressed using the Fourier identity

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\omega-\omega_0)t}\,\mathrm{d}t = 2\pi\,\delta(\omega-\omega_0). \tag{79}$$

Frequency-domain representation. Applying the Fourier transform to the real field (78), we obtain

$$\boxed{\hat{\mathbf{E}}_{\text{phys}}(\mathbf{r},\omega) = \frac{1}{2}\left[\mathbf{E}(\mathbf{r})\,\delta(\omega-\omega_0) + \mathbf{E}^*(\mathbf{r})\,\delta(\omega+\omega_0)\right].} \tag{80}$$

Physical meaning. This expression states that:

- the field has *support only at* $\omega = \pm\omega_0$,
- there is no broadband content,
- all energy is concentrated at a single temporal frequency,
- the negative-frequency term enforces reality of the time-domain field.

In contrast, a pulse or transient field would have a continuous frequency spectrum.

6.3 Single-frequency Maxwell equations

Projection onto a single frequency. Because Maxwell's equations are linear, each frequency component evolves independently. For a monochromatic field, we can therefore restrict attention to the ω_0 component alone.

This is equivalent to inserting the ansatz

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r})\, \mathrm{e}^{-\mathrm{i}\omega_0 t}, \qquad \mathbf{H}(\mathbf{r}, t) = \mathbf{H}(\mathbf{r})\, \mathrm{e}^{-\mathrm{i}\omega_0 t},$$

directly into the time-domain Maxwell equations.

Resulting equations. Time derivatives become algebraic:

$$\frac{\partial}{\partial t} \longrightarrow -\mathrm{i}\omega_0.$$

The Maxwell equations reduce to

$$\boxed{\begin{aligned} \nabla \cdot \mathbf{D}(\mathbf{r}) &= \rho(\mathbf{r}), \\ \nabla \cdot \mathbf{B}(\mathbf{r}) &= 0, \\ \nabla \times \mathbf{E}(\mathbf{r}) &= \mathrm{i}\omega_0 \mathbf{B}(\mathbf{r}), \\ \nabla \times \mathbf{H}(\mathbf{r}) &= \mathbf{j}(\mathbf{r}) - \mathrm{i}\omega_0 \mathbf{D}(\mathbf{r}). \end{aligned}} \tag{81}$$

Interpretation. These equations describe:

- steady-state fields oscillating at frequency ω_0,
- spatial field distributions determined by geometry and material properties,
- a boundary-value or eigenvalue problem rather than an initial-value problem.

Connection to earlier results. In source-free, homogeneous regions, Eq. (81) leads directly to:

- the vector Helmholtz equation,
- plane-wave and evanescent-wave solutions,
- modal equations for waveguides and cavities.

6.4 What monochromatic fields can and cannot describe

Captured physics.

- Wave propagation at a fixed wavelength,
- Resonant modes and standing waves,
- Phase accumulation and interference,
- Scattering at a single frequency.

Excluded physics.

- Transients and switching dynamics,
- Pulses and group velocity effects,
- Broadband dispersion in a single solve,
- Time-dependent modulation.

Thus, monochromatic fields form the foundation of frequency-domain photonics, while time-domain methods are required for dynamical phenomena.