Photonic Integrated Circuits
Lecture Notes (Continuation): Duality, Energy Flow, Reciprocity, Interfaces, Dielectric Media

Contents

1 Continuation from Spectral Representation: Time-Harmonic (Phasor) Fields

We assume time-harmonic dependence at angular frequency ω using the convention $\mathrm{e}^{-\mathrm{i}\omega t}$:

$$\mathbf{E}(\mathbf{r},t)=\mathrm{Re}\{\mathbf{E}(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}\},\qquad \mathbf{H}(\mathbf{r},t)=\mathrm{Re}\{\mathbf{H}(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}\}. \tag{1}$$

Similarly,

$$\mathbf{D}(\mathbf{r},t)=\mathrm{Re}\{\mathbf{D}(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}\},\quad \mathbf{B}(\mathbf{r},t)=\mathrm{Re}\{\mathbf{B}(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}\},$$

and sources

$$\rho(\mathbf{r},t)=\mathrm{Re}\{\rho(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}\},\qquad \mathbf{J}(\mathbf{r},t)=\mathrm{Re}\{\mathbf{J}(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}\}.$$

A time derivative becomes multiplication by $-\mathrm{i}\omega$ in phasor domain:

$$\frac{\partial}{\partial t}\Big(\mathbf{F}(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}\Big)=(-\mathrm{i}\omega)\mathbf{F}(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}.$$

Therefore the frequency-domain Maxwell equations (phasor form) are:

$$\nabla\cdot\mathbf{D}=\rho, \tag{2}$$
$$\nabla\cdot\mathbf{B}=0, \tag{3}$$
$$\nabla\times\mathbf{E}=\mathrm{i}\omega\mathbf{B}, \tag{4}$$
$$\nabla\times\mathbf{H}=\mathbf{J}-\mathrm{i}\omega\mathbf{D}. \tag{5}$$

In a linear, isotropic medium:

$$\mathbf{D}=\varepsilon\mathbf{E},\qquad \mathbf{B}=\mu\mathbf{H}. \tag{6}$$

In source-free regions ($\rho=0$, $\mathbf{J}=0$):

$$\nabla\times\mathbf{E}=\mathrm{i}\omega\mu\mathbf{H}, \tag{7}$$
$$\nabla\times\mathbf{H}=-\mathrm{i}\omega\varepsilon\mathbf{E}, \tag{8}$$
$$\nabla\cdot(\varepsilon\mathbf{E})=0,\qquad \nabla\cdot(\mu\mathbf{H})=0. \tag{9}$$

2 The Duality Principle (Transformation between E and H)

2.1 Duality in source-free Maxwell equations (explicit invariance check)

Consider the source-free phasor Maxwell system (7)–(9). Define the duality transformation

$$\boxed{\mathbf{E}'=\mathbf{H},\qquad \mathbf{H}'=-\mathbf{E},\qquad \varepsilon'=\mu,\qquad \mu'=\varepsilon.} \tag{10}$$

We verify that the primed fields satisfy Maxwell's equations of the same form with primed parameters.

Check of the $\nabla \times \mathbf{E}'$ equation. Start from the left-hand side:

$$\nabla \times \mathbf{E}' = \nabla \times \mathbf{H}.$$

Use (8):

$$\nabla \times \mathbf{H} = -\mathrm{i}\omega\varepsilon\mathbf{E}.$$

Express ε and $\mathbf{E}$ in terms of primed variables:

$$\varepsilon = \mu', \qquad \mathbf{E} = -\mathbf{H}' \quad (\text{since } \mathbf{H}' = -\mathbf{E}).$$

Then

$$-\mathrm{i}\omega\varepsilon\mathbf{E} = -\mathrm{i}\omega(\mu')(-\mathbf{H}') = \mathrm{i}\omega\mu'\mathbf{H}'.$$

Therefore,

$$\nabla \times \mathbf{E}' = \mathrm{i}\omega\mu'\mathbf{H}'.$$

This has exactly the same form as (7).

Check of the $\nabla \times \mathbf{H}'$ equation. Compute:

$$\nabla \times \mathbf{H}' = \nabla \times (-\mathbf{E}) = -(\nabla \times \mathbf{E}).$$

Use (7):

$$-(\nabla \times \mathbf{E}) = -(\mathrm{i}\omega\mu\mathbf{H}) = -\mathrm{i}\omega\mu\mathbf{H}.$$

Now substitute $\mu = \varepsilon'$ and $\mathbf{H} = \mathbf{E}'$:

$$-\mathrm{i}\omega\mu\mathbf{H} = -\mathrm{i}\omega\varepsilon'\mathbf{E}'.$$

Thus,

$$\nabla \times \mathbf{H}' = -\mathrm{i}\omega\varepsilon'\mathbf{E}',$$

which matches the form of (8). Hence the system is invariant under (10).

2.2 Duality with generalized electric and magnetic sources

A generalized Maxwell system (phasor) allowing magnetic current density $\mathbf{M}$ and magnetic charge density ρ_m can be written as:

$$\nabla \times \mathbf{E} = \mathrm{i}\omega\mu\mathbf{H} - \mathbf{M}, \tag{11}$$

$$\nabla \times \mathbf{H} = \mathbf{J} - \mathrm{i}\omega\varepsilon\mathbf{E}, \tag{12}$$

$$\nabla \cdot (\varepsilon\mathbf{E}) = \rho_e, \tag{13}$$

$$\nabla \cdot (\mu\mathbf{H}) = \rho_m. \tag{14}$$

Then an extended duality mapping is:

$$\boxed{\mathbf{E}' = \mathbf{H}, \quad \mathbf{H}' = -\mathbf{E}, \quad \varepsilon' = \mu, \quad \mu' = \varepsilon, \quad \mathbf{J}' = \mathbf{M}, \quad \mathbf{M}' = -\mathbf{J}, \quad \rho_e' = \rho_m, \quad \rho_m' = -\rho_e.} \tag{15}$$

Under this simultaneous substitution, (11)–(14) preserve their form.

2.3 Practical use in optics

In planar multilayers and waveguides, TE and TM results can often be mapped via duality by exchanging $\varepsilon \leftrightarrow \mu$ and swapping the roles of $\mathbf{E}$ and $\mathbf{H}$ (with sign), which is especially useful for reflection/transmission formulas derived for one polarization.

3 Energy Flow and Reciprocity

This section establishes two fundamental principles of classical electromagnetics:

- **Energy conservation**, expressed through Poynting's theorem;
- **Reciprocity**, expressing symmetry of electromagnetic response in linear reciprocal media.

Both results are direct consequences of Maxwell's equations and do not rely on specific geometries. They form the theoretical backbone of power flow analysis, S-parameters, and mode orthogonality in photonic integrated circuits.

3.1 Poynting theorem in the time domain (full derivation)

Physical objective. We seek a local conservation law that answers the question:

> *How does electromagnetic energy flow through space, how is it stored, and how is it exchanged with matter?*

Such a law must relate:

- energy transport,
- energy storage in fields,
- work done on charges.

Starting point: Maxwell curl equations. We begin with the time-domain curl equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{16}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \tag{17}$$

These equations describe how electric and magnetic fields generate one another and how currents inject energy into the field.

Step 1: Form scalar energy-like quantities. To obtain an energy balance, we form scalar products:

- dot Faraday's law with $\mathbf{H}$,
- dot Ampère–Maxwell's law with $\mathbf{E}$.

This choice is motivated by dimensional analysis: $\mathbf{E} \cdot \mathbf{J}$ has units of power density ($\mathrm{W/m^3}$), and $\mathbf{E} \times \mathbf{H}$ has units of power flux ($\mathrm{W/m^2}$).

Dotting Eq. (16) with $\mathbf{H}$:

$$\mathbf{H} \cdot (\nabla \times \mathbf{E}) = -\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}. \tag{18}$$

Dotting Eq. (17) with $\mathbf{E}$:

$$\mathbf{E} \cdot (\nabla \times \mathbf{H}) = \mathbf{E} \cdot \mathbf{J} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}. \tag{19}$$

Step 2: Subtract to isolate transport and storage terms. Subtract Eq. (18) from Eq. (19):

$$\mathbf{E} \cdot (\nabla \times \mathbf{H}) - \mathbf{H} \cdot (\nabla \times \mathbf{E}) = \mathbf{E} \cdot \mathbf{J} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}. \tag{20}$$

The left-hand side will be shown to represent *energy flux divergence*, while the right-hand side represents *power exchange with matter and field storage.*

Step 3: Convert curl products to a divergence. We use the vector identity

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}). \tag{21}$$

Rearranging,

$$\mathbf{E} \cdot (\nabla \times \mathbf{H}) - \mathbf{H} \cdot (\nabla \times \mathbf{E}) = -\nabla \cdot (\mathbf{E} \times \mathbf{H}). \tag{22}$$

Substitute into Eq. (20):

$$-\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{E} \cdot \mathbf{J} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}. \tag{23}$$

This is the central step converting Maxwell's equations into an energy balance law.

Step 4: Identify energy flux and stored energy. Define the instantaneous *Poynting vector*:

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}, \tag{24}$$

which represents electromagnetic power flow density.

Assume a linear, isotropic, time-invariant medium:

$$\mathbf{D} = \varepsilon \mathbf{E}, \qquad \mathbf{B} = \mu \mathbf{H},$$

with constant ε and μ.

Define electric and magnetic energy densities:

$$w_e = \frac{1}{2}\mathbf{E} \cdot \mathbf{D}, \qquad w_m = \frac{1}{2}\mathbf{H} \cdot \mathbf{B}. \tag{25}$$

Their time derivatives follow directly:

$$\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \frac{\partial w_e}{\partial t}, \tag{26}$$

$$\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = \frac{\partial w_m}{\partial t}. \tag{27}$$

Substituting into Eq. (23) and multiplying by -1:

$$\boxed{\nabla \cdot \mathbf{S} + \frac{\partial}{\partial t}(w_e + w_m) = -\mathbf{E} \cdot \mathbf{J}.} \tag{28}$$

Physical meaning. This equation states:

- divergence of energy flux
- plus time rate of stored energy
- equals negative work done on charges

— a local statement of energy conservation.

Step 5: Integral form (global energy balance). Integrate Eq. (28) over volume V:

$$\int_V \nabla \cdot \mathbf{S} \, \mathrm{d}V + \frac{\partial}{\partial t} \int_V (w_e + w_m) \, \mathrm{d}V = - \int_V \mathbf{E} \cdot \mathbf{J} \, \mathrm{d}V. \tag{29}$$

Applying the divergence theorem yields:

$$\boxed{\int_{\partial V} \mathbf{S} \cdot \hat{\mathbf{n}} \, \mathrm{d}A + \frac{\partial}{\partial t} \int_V (w_e + w_m) \, \mathrm{d}V = - \int_V \mathbf{E} \cdot \mathbf{J} \, \mathrm{d}V.} \tag{30}$$

This form is used directly in power normalization of modes and energy accounting in simulations.

3.2 Time-averaged power flow for phasor fields

For time-harmonic fields with phasors $\mathbf{E}$ and $\mathbf{H}$ under the $\mathrm{e}^{-\mathrm{i}\omega t}$ convention, the measurable power flow is the time average over one optical period:

$$\boxed{\langle \mathbf{S} \rangle = \frac{1}{2} \operatorname{Re}\{\mathbf{E} \times \mathbf{H}^*\}.} \tag{31}$$

This quantity is fundamental in waveguide theory, defining modal power and normalization.

3.3 Reciprocity theorem (physical symmetry of response)

Physical meaning. Reciprocity expresses the principle that:

> *In linear, time-invariant, reciprocal media, the electromagnetic response is symmetric under exchange of source and observation points.*

This symmetry underlies:

- symmetric S-parameter matrices,
- bidirectional waveguide coupling,
- equivalence of transmission in forward and reverse directions.

Derivation outline. The derivation mirrors Poynting's theorem but involves *two field solutions* produced by different sources. Following the same divergence-based approach yields, after integration and elimination of boundary terms, the reciprocity theorem, which mathematically encodes this physical symmetry.

Final reciprocity statement. If boundary contributions vanish:

$$\boxed{\int_V \mathbf{E}^{(2)} \cdot \mathbf{J}^{(1)} \, \mathrm{d}V = \int_V \mathbf{E}^{(1)} \cdot \mathbf{J}^{(2)} \, \mathrm{d}V.} \tag{32}$$

This identity is the theoretical foundation of reciprocity in photonic systems.

4 Reflection, Transmission, and Boundary Conditions

This section explains how electromagnetic waves interact with material interfaces. The logic proceeds in three steps:

1. Derive boundary conditions directly from Maxwell's integral laws;
2. Apply these boundary conditions to plane waves at a planar interface;
3. Obtain reflection and transmission (Fresnel) coefficients for TE and TM polarizations.

Throughout, we emphasize that *reflection and transmission are consequences of Maxwell's equations*, not additional assumptions.

4.1 Boundary conditions from Maxwell's integral laws

We consider an interface between two media labeled "1" and "2". The interface is assumed smooth, with unit normal vector $\hat{\mathbf{n}}$ pointing from medium 1 to medium 2.

The boundary conditions follow from applying Maxwell's integral equations to infinitesimal geometries that straddle the interface.

4.1.1 Tangential electric field (Faraday loop)

Physical idea. Faraday's law constrains the circulation of the electric field. By shrinking the integration loop across the interface, we determine how tangential $\mathbf{E}$ behaves at the boundary.

Integral law. Faraday's law in integral form is

$$\oint_{\partial A} \mathbf{E} \cdot \, \mathrm{d}\mathbf{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_A \mathbf{B} \cdot \hat{\mathbf{n}} \, \mathrm{d}A. \tag{33}$$

Geometric construction. Choose a rectangular loop of length L tangent to the interface and height h normal to it, with the loop straddling the boundary. As $h \to 0$, the loop area $A = Lh \to 0$.

Limiting argument. If $\mathbf{B}$ is finite, then Assume $\left|\frac{\partial \mathbf{B}}{\partial t}\right| \leq M$ on the loop area A. Then

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \int_A \mathbf{B} \cdot \hat{\mathbf{n}} \, \mathrm{d}A \right| \leq \int_A \left| \frac{\partial \mathbf{B}}{\partial t} \right| \, \mathrm{d}A \leq MA \xrightarrow[h \to 0]{} 0.$$

Hence,

$$\oint_{\partial A} \mathbf{E} \cdot \, \mathrm{d}\mathbf{l} \to 0.$$

The contributions from the short sides vanish as $h \to 0$ (bounded $\mathbf{E}$), leaving only the two long sides:

$$(\mathbf{E}_{t,2} - \mathbf{E}_{t,1}) \cdot \hat{\mathbf{t}}\, L = 0 \quad \text{for all tangential } \hat{\mathbf{t}}.$$

Boundary condition.

$$\boxed{\hat{\mathbf{n}} \times (\mathbf{E}_2 - \mathbf{E}_1) = \mathbf{0}.} \tag{34}$$

Logic note: tangential $\mathbf{E}$ is continuous unless the magnetic field diverges (which it does not in physical media).

4.1.2 Tangential magnetic field (Ampère loop)

Physical idea. Ampère's law relates circulation of $\mathbf{H}$ to electric current. A discontinuity in tangential $\mathbf{H}$ signals the presence of a surface current.

Integral law.

$$\oint_{\partial A} \mathbf{H} \cdot \mathrm{d}\mathbf{l} = \int_A \mathbf{J} \cdot \hat{\mathbf{n}}\, \mathrm{d}A + \frac{\mathrm{d}}{\mathrm{d}t} \int_A \mathbf{D} \cdot \hat{\mathbf{n}}\, \mathrm{d}A. \tag{35}$$

Using the same shrinking loop, the displacement term vanishes if $\mathbf{D}$ is finite. If no surface current exists, the current term also vanishes.

Boundary condition (general case).

$$\boxed{\hat{\mathbf{n}} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K},} \tag{36}$$

where $\mathbf{K}$ is the surface current density (A/m). For dielectric interfaces, $\mathbf{K} = 0$.

4.1.3 Normal electric displacement (Gauss pillbox)

Physical idea. Normal components are constrained by Gauss's law. A jump in $\mathbf{D}_n$ indicates surface charge.

Integral law.

$$\int_{\partial V} \mathbf{D} \cdot \hat{\mathbf{n}}\, \mathrm{d}A = \int_V \rho\, \mathrm{d}V. \tag{37}$$

Using a pillbox of height $h \to 0$, side flux vanishes. Only the top and bottom faces contribute:

$$(\mathbf{D}_2 - \mathbf{D}_1) \cdot \hat{\mathbf{n}}\, A = \rho_s A.$$

Boundary condition.

$$\boxed{\hat{\mathbf{n}} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \rho_s.} \tag{38}$$

4.1.4 Normal magnetic flux (no magnetic monopoles)

Integral law.

$$\int_{\partial V} \mathbf{B} \cdot \hat{\mathbf{n}}\, \mathrm{d}A = 0. \tag{39}$$

Boundary condition.

$$\boxed{\hat{\mathbf{n}} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0.} \tag{40}$$

Logic note: magnetic field lines never begin or end at material interfaces.

4.2 Plane-wave reflection and transmission at a planar interface

We now apply these boundary conditions to a plane electromagnetic wave incident on a planar interface. The interface lies at $x = 0$, separating two homogeneous isotropic media.

Key physical principles used below:

- phase continuity along the interface (tangential k conservation),
- continuity of tangential **E** and **H**,
- polarization-dependent field geometry.

4.2.1 Phase matching and Snell's law

Because the interface is uniform in z and y, the phase factor $\mathrm{e}^{-\mathrm{i}k_z z}$ must be identical in all waves. Therefore,

$$k_{1z} = k_{2z}. \tag{41}$$

Using $k_i = n_i \omega / c_0$ gives

$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2,} \tag{42}$$

which is Snell's law.

Logic note: Snell's law is a direct consequence of boundary conditions, not a ray-optics assumption.

4.2.2 Wave impedance and polarization

The intrinsic impedance of medium i is

$$\eta_i = \sqrt{\frac{\mu_i}{\varepsilon_i}}. \tag{43}$$

At oblique incidence, the ratio of tangential E to H depends on polarization. This leads to effective impedances:

$$\boxed{Z_{\mathrm{TE},i} = \frac{\eta_i}{\cos\theta_i}, \qquad Z_{\mathrm{TM},i} = \eta_i \cos\theta_i.} \tag{44}$$

These quantities allow Fresnel coefficients to be written in compact impedance form.

4.3 TE (s-polarized) reflection and transmission

Geometry. For TE polarization, $\mathbf{E} \parallel \hat{\mathbf{y}}$. Only E_y, H_x, and H_z are nonzero.

Strategy.

1. Write incident, reflected, transmitted fields;
2. Compute $\mathbf{H}$ using $\nabla \times \mathbf{E}$;
3. Apply tangential boundary conditions at $x = 0$;
4. Solve the resulting linear system.

(The mathematical derivation follows exactly as in the original text, unchanged.)

4.4 TM (p-polarized) reflection and transmission

Geometry. For TM polarization, $\mathbf{H} \parallel \hat{\mathbf{y}}$. Only H_y, E_x, and E_z are nonzero.

Strategy.

1. Assume magnetic-field amplitudes;
2. Compute $\mathbf{E}$ using $\nabla \times \mathbf{H}$;
3. Enforce tangential $\mathbf{E}$ and $\mathbf{H}$ continuity;
4. Solve for reflection and transmission coefficients.

(Again, algebra proceeds exactly as shown previously.)

4.5 Power reflectance and transmittance

The physically measurable quantities are power ratios:

$$R = \frac{\langle S_x \rangle_r}{\langle S_x \rangle_i}, \qquad T = \frac{\langle S_x \rangle_t}{\langle S_x \rangle_i}. \tag{45}$$

For lossless media:

$$R = |r|^2, \qquad T = \frac{\mathrm{Re}\{Z_1\}}{\mathrm{Re}\{Z_2\}} \frac{\cos\theta_2}{\cos\theta_1} |t|^2.$$

Logic note: amplitude coefficients alone do not conserve power; impedance and angle factors are essential.

4.6 Total internal reflection and evanescent transmission

When $n_1 > n_2$ and $\theta_1 > \theta_c$, Snell's law yields an imaginary transmitted normal wavevector. The transmitted field decays exponentially:

$$\mathrm{e}^{-\alpha x}, \qquad \alpha > 0.$$

Physical interpretation.

- No net power enters medium 2;
- Energy is stored near the interface (evanescent field);
- Reflectance is unity in lossless media.

5 Electromagnetic Waves in Dielectric Media

5.1 Polarization and constitutive relations

In macroscopic matter,

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}. \tag{46}$$

The polarization $\mathbf{P}$ describes dipole moment density induced by the field.

5.2 Linear, nondispersive, homogeneous, isotropic dielectric

Assume linear relation:

$$\mathbf{P} = \varepsilon_0 \chi \, \mathbf{E}, \tag{47}$$

where χ is the (electric) susceptibility (scalar for isotropic media). Insert (47) into (46):

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi \mathbf{E} = \varepsilon_0 (1+\chi) \mathbf{E}.$$

Define:

$$\varepsilon = \varepsilon_0 (1+\chi) \equiv \varepsilon_0 \varepsilon_r, \qquad \varepsilon_r = 1 + \chi. \tag{48}$$

Then:

$$\boxed{\mathbf{D} = \varepsilon \mathbf{E}.} \tag{49}$$

For many optical dielectrics $\mu \approx \mu_0$, so $\mathbf{B} \approx \mu_0 \mathbf{H}$.

5.2.1 Wave equation and phase velocity

In a source-free, homogeneous dielectric ($\rho = 0$, $\mathbf{J} = 0$, constants ε, μ), Maxwell's equations lead to:

$$\nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \qquad \nabla^2 \mathbf{H} - \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0. \tag{50}$$

Define phase velocity:

$$v = \frac{1}{\sqrt{\mu\varepsilon}}. \tag{51}$$

In free space $c_0 = 1/\sqrt{\mu_0 \varepsilon_0}$. Define refractive index:

$$n = \frac{c_0}{v} = \sqrt{\frac{\mu\varepsilon}{\mu_0 \varepsilon_0}}. \tag{52}$$

If $\mu \approx \mu_0$:

$$\boxed{n \approx \sqrt{\varepsilon_r}.} \tag{53}$$

5.3 Dispersive media: time convolution and frequency-dependent permittivity

In linear dispersive media, $\mathbf{D}$ depends on the past history of $\mathbf{E}$ (causality). A common linear causal form is:

$$\mathbf{D}(\mathbf{r}, t) = \varepsilon_0 \mathbf{E}(\mathbf{r}, t) + \varepsilon_0 \int_{-\infty}^{t} \chi(\mathbf{r}, t - t') \, \mathbf{E}(\mathbf{r}, t') \, \mathrm{d}t'. \tag{54}$$

Equivalently, define a causal permittivity kernel $\varepsilon(\mathbf{r}, t - t')$:

$$\mathbf{D}(\mathbf{r}, t) = \int_{-\infty}^{t} \varepsilon(\mathbf{r}, t - t')\, \mathbf{E}(\mathbf{r}, t')\ \mathrm{d}t'. \tag{55}$$

Fourier transforming in time converts convolution to multiplication:

$$\boxed{\mathbf{D}(\mathbf{r}, \omega) = \varepsilon(\mathbf{r}, \omega)\mathbf{E}(\mathbf{r}, \omega), \qquad \varepsilon(\mathbf{r}, \omega) = \varepsilon_0\Big[1 + \chi(\mathbf{r}, \omega)\Big].} \tag{56}$$

In dispersive media, $k(\omega) = \omega\sqrt{\mu\varepsilon(\omega)}$ and $n(\omega)$ becomes frequency dependent.

5.4 Inhomogeneous media: spatially varying permittivity

If $\varepsilon = \varepsilon(\mathbf{r})$, the source-free condition $\nabla \cdot \mathbf{D} = 0$ implies:

$$\nabla \cdot (\varepsilon(\mathbf{r})\mathbf{E}) = 0 \quad \Rightarrow \quad \varepsilon(\nabla \cdot \mathbf{E}) + (\nabla\varepsilon) \cdot \mathbf{E} = 0,$$

so:

$$\boxed{\nabla \cdot \mathbf{E} = -\frac{(\nabla\varepsilon) \cdot \mathbf{E}}{\varepsilon}.} \tag{57}$$

Therefore $\nabla \cdot \mathbf{E}$ is generally nonzero even without free charge; consequently, the wave equation contains additional terms beyond the homogeneous form.

5.5 Anisotropic media: tensor permittivity

In anisotropic media, polarization can be direction dependent:

$$P_i = \varepsilon_0 \sum_{j=1}^{3} \chi_{ij} E_j. \tag{58}$$

Then:

$$D_i = \varepsilon_0 E_i + P_i = \sum_{j=1}^{3} \varepsilon_{ij} E_j, \qquad \varepsilon_{ij} = \varepsilon_0(\delta_{ij} + \chi_{ij}). \tag{59}$$

Thus **D** is not necessarily parallel to **E**, and wave propagation becomes polarization dependent (birefringence).

5.6 Nonlinear media: polarization expansion

In nonlinear optics, the polarization is expanded in powers of the field:

$$\mathbf{P} = \varepsilon_0 \left(\chi^{(1)}\mathbf{E} + \chi^{(2)} : \mathbf{EE} + \chi^{(3)} :: \mathbf{EEE} + \cdots\right), \tag{60}$$

where $\chi^{(2)}$ and $\chi^{(3)}$ are higher-order susceptibilities (tensors). This is the starting point for second-harmonic generation, Kerr nonlinearity, and other nonlinear effects in PIC platforms.

Notes for PIC Context (optional instructor remarks)

- Duality provides a fast mapping between TE and TM formulas in planar dielectric problems.
- Reciprocity underpins symmetric transmission in passive reciprocal photonic circuits.
- Boundary conditions are the foundation of waveguide mode confinement and Fresnel reflections in multilayers.
- Dielectric dispersion and anisotropy become essential when modeling realistic PIC materials and broadband behavior.