Photonic Integrated Circuits — Lecture Notes (Detailed)
Polarization & TEM/TE/TM Fields; Elementary EM Waves; Absorption/Dispersion; Scattering; Pulse Propagation

Contents

1 Conventions and Starting Point (Phasors, Maxwell, Constitutive Laws)

1.1 Time-harmonic convention

We adopt the phasor convention $\mathrm{e}^{-\mathrm{i}\omega t}$. Physical fields are the real parts:

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\{\mathbf{E}(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}\}, \qquad \mathbf{H}(\mathbf{r},t) = \mathrm{Re}\{\mathbf{H}(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}\}. \tag{1}$$

Thus $\partial/\partial t \mapsto -\mathrm{i}\omega$ for phasors.

1.2 Maxwell equations in phasor form

$$\nabla \cdot \mathbf{D} = \rho, \tag{2}$$
$$\nabla \cdot \mathbf{B} = 0, \tag{3}$$
$$\nabla \times \mathbf{E} = \mathrm{i}\omega \mathbf{B}, \tag{4}$$
$$\nabla \times \mathbf{H} = \mathbf{J} - \mathrm{i}\omega \mathbf{D}. \tag{5}$$

Linear isotropic constitutive relations:

$$\mathbf{D} = \varepsilon \mathbf{E}, \qquad \mathbf{B} = \mu \mathbf{H}. \tag{6}$$

In many optical dielectrics $\mu \approx \mu_0$; we keep μ general until specializing.

1.3 Source-free homogeneous region

In a homogeneous, source-free region:

$$\rho = 0, \quad \mathbf{J} = 0, \quad \varepsilon = \text{const}, \quad \mu = \text{const}, \tag{7}$$

Maxwell reduces to

$$\nabla \times \mathbf{E} = \mathrm{i}\omega\mu \mathbf{H}, \tag{8}$$
$$\nabla \times \mathbf{H} = -\mathrm{i}\omega\varepsilon \mathbf{E}, \tag{9}$$
$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{H} = 0. \tag{10}$$

2 Polarization and TEM Fields (TE, TM Solutions)

This section develops the fundamental structure of electromagnetic plane waves and introduces the TE, TM, and TEM classifications. The results here will be reused directly when reducing Maxwell's equations to waveguide eigenvalue problems.

2.1 Plane waves: transversality, dispersion, impedance, and polarization

2.1.1 Plane-wave ansatz and differential operators

Physical assumption. In a homogeneous, source-free medium, we seek solutions that propagate without change of shape. This motivates the plane-wave ansatz, where spatial dependence is purely exponential.

Assume time-harmonic phasors of the form

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}, \qquad \mathbf{H}(\mathbf{r}) = \mathbf{H}_0\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}, \tag{11}$$

where $\mathbf{k}$ is the wavevector and $\mathbf{E}_0$, $\mathbf{H}_0$ are constant complex amplitudes.

Key operator identities. For any constant vector $\mathbf{F}_0$,

$$\nabla \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} = \mathrm{i}\mathbf{k}\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}},$$

$$\nabla \times (\mathbf{F}_0 \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}) = \mathrm{i}(\mathbf{k}\times\mathbf{F}_0)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}, \qquad \nabla \cdot (\mathbf{F}_0 \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}) = \mathrm{i}(\mathbf{k}\cdot\mathbf{F}_0)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}.$$

These identities convert Maxwell's differential equations into algebraic relations between $\mathbf{E}_0$, $\mathbf{H}_0$, and $\mathbf{k}$.

2.1.2 Relation between field amplitudes $\mathbf{E}_0$ and $\mathbf{H}_0$

Insert the plane-wave ansatz into Faraday's law (source-free):

$$\nabla \times \mathbf{E} = \mathrm{i}\omega\mu\mathbf{H}.$$

Using the curl identity above,

$$\mathrm{i}(\mathbf{k}\times\mathbf{E}_0)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} = \mathrm{i}\omega\mu\mathbf{H}_0\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}.$$

Cancel $\mathrm{i}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}$ to obtain

$$\boxed{\mathbf{k}\times\mathbf{E}_0 = \omega\mu\,\mathbf{H}_0 \quad \Rightarrow \quad \mathbf{H}_0 = \frac{1}{\omega\mu}\mathbf{k}\times\mathbf{E}_0.} \tag{12}$$

Logic note: The magnetic field is not independent; it is generated by spatial rotation of the electric field.

2.1.3 Transversality and dispersion relation

From Gauss's law in a source-free region, $\nabla\cdot\mathbf{E} = 0$:

$$\mathrm{i}(\mathbf{k}\cdot\mathbf{E}_0)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} = 0 \quad \Rightarrow \quad \mathbf{k}\cdot\mathbf{E}_0 = 0.$$

Similarly, $\mathbf{k}\cdot\mathbf{H}_0 = 0$.

$$\boxed{\mathbf{k}\cdot\mathbf{E}_0 = 0, \qquad \mathbf{k}\cdot\mathbf{H}_0 = 0.} \tag{13}$$

Physical meaning: Plane electromagnetic waves are strictly transverse: both $\mathbf{E}$ and $\mathbf{H}$ lie in the plane perpendicular to the direction of propagation.

Now insert the plane-wave form into Ampère–Maxwell's law:

$$\nabla \times \mathbf{H} = -\mathrm{i}\omega\varepsilon\mathbf{E}.$$

Using (12):

$$\mathbf{k}\times\left(\frac{1}{\omega\mu}\mathbf{k}\times\mathbf{E}_0\right) = -\omega\varepsilon\mathbf{E}_0.$$

Apply the vector identity $\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\mathbf{a} \cdot \mathbf{b}) - \mathbf{b}(\mathbf{a} \cdot \mathbf{a})$, and use $\mathbf{k} \cdot \mathbf{E}_0 = 0$:

$$-\mathbf{E}_0 k^2 = -\omega^2 \mu\varepsilon \mathbf{E}_0.$$

Hence,

$$\boxed{k = \omega\sqrt{\mu\varepsilon}, \qquad v_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}}.} \tag{14}$$

Logic note: The wavevector magnitude is fixed by material properties alone. This is the electromagnetic origin of refractive index.

2.1.4 Wave impedance

From (12) and transversality,

$$|\mathbf{H}_0| = \frac{1}{\omega\mu}|\mathbf{k} \times \mathbf{E}_0| = \frac{k}{\omega\mu}|\mathbf{E}_0|.$$

Using $k = \omega\sqrt{\mu\varepsilon}$,

$$|\mathbf{H}_0| = \sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}_0|.$$

Thus,

$$\boxed{|\mathbf{E}_0| = \eta|\mathbf{H}_0|, \qquad \eta = \sqrt{\frac{\mu}{\varepsilon}}.} \tag{15}$$

Physical meaning: η sets the ratio of electric to magnetic energy in a propagating wave.

2.1.5 Polarization ellipse

Because $\mathbf{E}_0$ is transverse, it can be decomposed into two orthonormal directions $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2 \perp \hat{\mathbf{k}}$:

$$\mathbf{E}_0 = E_1\hat{\mathbf{e}}_1 + E_2\hat{\mathbf{e}}_2, \qquad E_j = |E_j|e^{i\phi_j}.$$

The physical electric field is

$$\mathbf{E}_{\text{phys}}(t) = |E_1|\cos(\omega t - \mathbf{k} \cdot \mathbf{r} + \phi_1)\hat{\mathbf{e}}_1 + |E_2|\cos(\omega t - \mathbf{k} \cdot \mathbf{r} + \phi_2)\hat{\mathbf{e}}_2.$$

Eliminating time yields an ellipse traced by the field tip. Special cases:

- Linear polarization: $\phi_2 - \phi_1 = 0, \pi$,
- Circular polarization: $|E_1| = |E_2|$ and $\phi_2 - \phi_1 = \pm\pi/2$,
- General elliptical polarization otherwise.

Linear polarization occurs when the two transverse components oscillate in phase or antiphase, so that their ratio is time-independent and the field direction is fixed.

2.2 TEM, TE, and TM field classifications

Assume predominant propagation along $+\hat{\mathbf{z}}$. Decompose fields into transverse and longitudinal parts:

$$\mathbf{E} = \mathbf{E}_t + \hat{\mathbf{z}}E_z, \qquad \mathbf{H} = \mathbf{H}_t + \hat{\mathbf{z}}H_z.$$

Definitions:

$$\text{TEM: } E_z = 0, \quad H_z = 0, \tag{16}$$

$$\text{TE: } E_z = 0, \quad H_z \neq 0, \tag{17}$$

$$\text{TM: } H_z = 0, \quad E_z \neq 0. \tag{18}$$

Logic note: Exact TEM waves require at least two conductors; hence they do not exist in hollow dielectric guides. Pure TE/TM modes arise only in special geometries (e.g. slab waveguides); otherwise modes are hybrid.

2.3 Exact TE/TM reduction in 1D stratified dielectrics

Assumptions:

- Material varies only in x: $\varepsilon = \varepsilon(x)$, $\mu = \mu(x)$,
- Invariance in y,
- z-propagation: $\partial/\partial z \to \mathrm{i}\beta$.

This reduces the vector Maxwell equations to scalar second-order ODEs.

2.3.1 TE polarization: scalar equation for $E_y(x)$

Assume

$$\mathbf{E}(x,z) = \hat{\mathbf{y}}\, E_y(x)\mathrm{e}^{\mathrm{i}\beta z}. \tag{19}$$

Following the steps shown earlier yields

$$\boxed{\frac{\mathrm{d}^2 E_y}{\mathrm{d}x^2} + \left(\omega^2\mu\varepsilon - \beta^2\right) E_y = 0.} \tag{20}$$

Boundary conditions at a dielectric interface:

$$\boxed{E_y \text{ continuous}, \qquad \frac{1}{\mu}\frac{\mathrm{d}E_y}{\mathrm{d}x} \text{ continuous.}} \tag{21}$$

2.3.2 TM polarization: scalar equation for $H_y(x)$

Assume

$$\mathbf{H}(x,z) = \hat{\mathbf{y}}\, H_y(x)\mathrm{e}^{\mathrm{i}\beta z}. \tag{22}$$

This yields

$$\boxed{\frac{\mathrm{d}^2 H_y}{\mathrm{d}x^2} + \left(\omega^2\mu\varepsilon - \beta^2\right) H_y = 0.} \tag{23}$$

Boundary conditions:

$$\boxed{H_y \text{ continuous}, \qquad \frac{1}{\varepsilon}\frac{\mathrm{d}H_y}{\mathrm{d}x} \text{ continuous.}} \tag{24}$$

2.3.3 Propagating vs. evanescent transverse behavior

Define

$$k_x^2 = \omega^2 \mu \varepsilon - \beta^2.$$

Then

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + k_x^2 \Psi = 0, \qquad \Psi = E_y \text{ (TE) or } H_y \text{ (TM)}.$$

- $k_x^2 > 0$: oscillatory (propagating),
- $k_x^2 < 0$: evanescent decay.

This distinction underlies dielectric confinement and all guided-wave optics in PICs.

3 Elementary Electromagnetic Waves

In photonics and electromagnetics, most fields encountered in practice can be understood as superpositions or limits of three fundamental wave families: *plane waves*, *spherical (dipole) waves*, and *Gaussian beams*. Each represents a different symmetry and physical situation.

3.1 Plane waves (review) and time-averaged energy flow

Plane waves describe fields that propagate without diffraction in a homogeneous medium. They are the local building blocks of more complicated wavefields.

Time-averaged energy flow. For time-harmonic fields, the physically relevant power flow is given by the time-averaged Poynting vector:

$$\langle \mathbf{S} \rangle = \frac{1}{2} \mathrm{Re}\{\mathbf{E} \times \mathbf{H}^*\}. \tag{25}$$

Logic: Instantaneous power oscillates rapidly at optical frequencies. Averaging over one optical period removes these oscillations and yields the net energy flow.

For a lossless plane wave:

- $\langle \mathbf{S} \rangle$ is parallel to the wavevector $\mathbf{k}$,
- energy propagates in the same direction as phase fronts.

The magnitude is

$$\langle S \rangle = \frac{1}{2} \frac{|\mathbf{E}_0|^2}{\eta} = \frac{1}{2} \eta |\mathbf{H}_0|^2,$$

where $\eta = \sqrt{\mu/\varepsilon}$ is the wave impedance of the medium.

3.2 Spherical waves and radiating dipoles

Plane waves are idealized solutions with infinite transverse extent. Localized sources instead generate *spherical waves.*

3.2.1 Outgoing spherical wave from the Helmholtz equation

In a homogeneous medium, the scalar Helmholtz equation reads

$$(\nabla^2 + k^2)u(\mathbf{r}) = 0. \tag{26}$$

A fundamental outgoing solution is

$$u(\mathbf{r}) = \frac{\mathrm{e}^{\mathrm{i}kr}}{r}, \qquad r = |\mathbf{r}|. \tag{27}$$

Logic:

- The factor $\mathrm{e}^{\mathrm{i}kr}$ represents outward-propagating phase fronts.
- The $1/r$ decay reflects conservation of energy over expanding spherical surfaces.

More rigorously, the Green function satisfies

$$(\nabla^2 + k^2)G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}'),$$

with solution

$$G(\mathbf{r}, \mathbf{r}') = \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|}. \tag{28}$$

Logic: This Green function represents the field generated by a point source and is the basis for all radiation and scattering solutions.

3.2.2 Oscillating electric dipole: far-field structure

An oscillating electric dipole

$$\mathbf{p}(t) = \mathrm{Re}\{\mathbf{p}_0\,\mathrm{e}^{-\mathrm{i}\omega t}\}$$

is the simplest radiating electromagnetic source.

In the radiation (far-field) zone, defined by $kr \gg 1$:

- fields are transverse to the radial direction $\hat{\mathbf{r}}$,
- field amplitudes scale as $1/r$,
- electric and magnetic fields are mutually orthogonal.

For a dipole oriented along $\hat{\mathbf{z}}$,

$$|\mathbf{E}_{\mathrm{rad}}| \propto \frac{\omega^2|p_0|}{r}\sin\theta, \qquad \mathbf{H}_{\mathrm{rad}} \approx \frac{1}{\eta}\,\hat{\mathbf{r}} \times \mathbf{E}_{\mathrm{rad}}.$$

Logic: The $\sin\theta$ angular dependence produces a toroidal (donut-shaped) radiation pattern, which appears universally in dipole radiation and Rayleigh scattering.

3.3 Gaussian beams and the paraxial approximation

Plane waves do not diffract, while spherical waves diverge strongly. Gaussian beams describe real laser beams that diffract slowly.

3.3.1 From Helmholtz to the paraxial wave equation

Start from the scalar Helmholtz equation:

$$\nabla^2 U + k^2 U = 0.$$

Assume a slowly varying envelope:

$$U(x, y, z) = \psi(x, y, z)\, \mathrm{e}^{\mathrm{i}kz}.$$

Logic: The rapid oscillation $\mathrm{e}^{\mathrm{i}kz}$ accounts for forward propagation, while ψ captures beam spreading and curvature.

Substituting into Helmholtz gives

$$\nabla_\perp^2 \psi + \frac{\partial^2 \psi}{\partial z^2} + 2\mathrm{i}k \frac{\partial \psi}{\partial z} = 0.$$

The paraxial approximation assumes that the wave propagates mainly along one preferred axis (here z), and that any variation of the field along this axis is slow compared to the optical wavelength.

The *paraxial approximation* assumes

$$\left| \frac{\partial^2 \psi}{\partial z^2} \right| \ll \left| k \frac{\partial \psi}{\partial z} \right|,$$

leading to

$$2\mathrm{i}k \frac{\partial \psi}{\partial z} + \nabla_\perp^2 \psi = 0. \tag{29}$$

The term $k\partial_z \psi$ comes from forward propagation. The term $\partial_z^2 \psi$ represents rapid longitudinal variation or backscattering. For beams that, are not tightly focused, do not bend sharply, and propagate mostly forward, the second longitudinal derivative is much smaller and can be neglected.

3.3.2 Fundamental Gaussian beam

A fundamental solution of the paraxial equation is

$$\psi(x, y, z) = \psi_0 \frac{w_0}{w(z)} \exp\left(-\frac{x^2 + y^2}{w^2(z)} \right) \exp\left(-\mathrm{i} \frac{k(x^2 + y^2)}{2R(z)} \right) \exp(\mathrm{i}\zeta(z)), \tag{30}$$

where

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_R} \right)^2}, \quad R(z) = z \left[1 + \left(\frac{z_R}{z} \right)^2 \right],$$

$$\zeta(z) = \arctan\left(\frac{z}{z_R} \right), \quad z_R = \frac{\pi w_0^2}{\lambda}.$$

Logic:

- $w(z)$ describes beam spreading,
- $R(z)$ gives wavefront curvature,
- $\zeta(z)$ is the Gouy phase shift.

For full electromagnetic beams, the condition $\nabla \cdot \mathbf{E} = 0$ introduces a small longitudinal field component. Its magnitude is of order

$$\theta_{\mathrm{div}} \sim \frac{w_0}{z_R},$$

and vanishes in the plane-wave limit.

4 Absorption and Dispersion

We derive absorption and dispersion from complex permittivity and causality, then connect to group velocity and pulse effects.

4.1 Causality ⇒ complex susceptibility and Kramers–Kronig

In a linear time-invariant dispersive medium, the polarization responds causally:

$$\mathbf{P}(t) = \varepsilon_0 \int_{-\infty}^{t} \chi(t-t')\mathbf{E}(t')\, \mathrm{d}t', \qquad \chi(t<0) = 0. \tag{31}$$

Then $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$ gives in frequency domain:

$$\boxed{\mathbf{D}(\omega) = \varepsilon(\omega)\mathbf{E}(\omega), \qquad \varepsilon(\omega) = \varepsilon_0[1+\chi(\omega)].} \tag{32}$$

Causality implies analyticity of $\chi(\omega)$ in the upper-half complex ω plane, yielding Kramers–Kronig relations between real and imaginary parts [2]:

$$\boxed{\chi'(\omega) = \frac{1}{\pi}\,\mathcal{P}\int_{-\infty}^{\infty} \frac{\chi''(\omega')}{\omega'-\omega}\, \mathrm{d}\omega',} \tag{33}$$

$$\boxed{\chi''(\omega) = -\frac{1}{\pi}\,\mathcal{P}\int_{-\infty}^{\infty} \frac{\chi'(\omega')}{\omega'-\omega}\, \mathrm{d}\omega'.} \tag{34}$$

Thus dispersion and absorption are fundamentally linked.

4.2 Complex refractive index and attenuation (Beer–Lambert)

Let

$$\varepsilon(\omega) = \varepsilon'(\omega) + \mathrm{i}\varepsilon''(\omega), \qquad \tilde{n}(\omega) = n(\omega) + \mathrm{i}\kappa(\omega), \tag{35}$$

with (for $\mu \approx \mu_0$) $\tilde{n}^2 \approx \varepsilon/\varepsilon_0$. Then the plane-wave propagation constant is

$$k(\omega) = k_0\tilde{n}(\omega) = \frac{\omega}{c_0}\left[n(\omega) + \mathrm{i}\kappa(\omega)\right].$$

A field factor $\mathrm{e}^{\mathrm{i}kz}$ becomes

$$\mathrm{e}^{\mathrm{i}kz} = \mathrm{e}^{\mathrm{i}k_0 n z}\,\mathrm{e}^{-k_0\kappa z}.$$

Hence *field amplitude* decays as $\mathrm{e}^{-\alpha z}$ with

$$\boxed{\alpha(\omega) = k_0\kappa(\omega) = \frac{\omega}{c_0}\kappa(\omega).} \tag{36}$$

Intensity $I \propto |\mathbf{E}|^2$ therefore obeys Beer–Lambert:

$$\boxed{I(z) = I(0)\,\mathrm{e}^{-2\alpha z} = I(0)\,\mathrm{e}^{-\alpha_I z}, \qquad \alpha_I = 2\alpha.} \tag{37}$$

4.3 Phase velocity, group velocity, and group index

Define the (real) phase constant

$$\beta(\omega) = \mathrm{Re}\{k(\omega)\} = \frac{\omega}{c_0} n(\omega). \tag{38}$$

Phase velocity:

$$v_p = \frac{\omega}{\beta} = \frac{c_0}{n(\omega)}.$$

Group velocity:

$$v_g(\omega) = \left(\frac{\mathrm{d}\beta}{\mathrm{d}\omega}\right)^{-1}. \tag{39}$$

Compute $\mathrm{d}\beta/\,\mathrm{d}\omega$ from (38):

$$\frac{\mathrm{d}\beta}{\mathrm{d}\omega} = \frac{1}{c_0}\left[n(\omega) + \omega \frac{\mathrm{d}n}{\mathrm{d}\omega}\right].$$

Therefore

$$\boxed{v_g(\omega) = \frac{c_0}{n(\omega) + \omega \frac{\mathrm{d}n}{\mathrm{d}\omega}}, \qquad n_g(\omega) = \frac{c_0}{v_g} = n(\omega) + \omega \frac{\mathrm{d}n}{\mathrm{d}\omega}.} \tag{40}$$

5 Scattering of Electromagnetic Waves (Born, Rayleigh, Mie)

Scattering describes how electromagnetic waves are redistributed in space due to *inhomogeneities* in material properties. In photonics, scattering determines losses, radiation, sensing, and near-field interactions.

We develop a natural hierarchy of models:

- **Born approximation**: weak scattering, single interaction with the object.
- **Rayleigh scattering**: electrically small particles, dipole response.
- **Mie scattering**: exact solution for a sphere of arbitrary size.

5.1 Material inhomogeneity as an effective scattering source

Assume:

- No free charges or currents in the region.
- Nonmagnetic media: $\mu = \mu_0$ everywhere.
- A homogeneous background with a localized perturbation.

Write the permittivity as

$$\varepsilon(\mathbf{r}) = \varepsilon_b + \Delta\varepsilon(\mathbf{r}), \qquad \Delta\varepsilon(\mathbf{r}) \text{ localized in space.}$$

Using the frequency-domain Maxwell equations, the electric field satisfies the vector wave equation

$$\nabla \times \nabla \times \mathbf{E} - k_0^2 \varepsilon_r(\mathbf{r})\mathbf{E} = 0, \qquad \varepsilon_r(\mathbf{r}) = \frac{\varepsilon(\mathbf{r})}{\varepsilon_0}. \tag{41}$$

Split the relative permittivity into background and perturbation:

$$\varepsilon_r(\mathbf{r}) = \varepsilon_{rb} + \Delta\varepsilon_r(\mathbf{r}).$$

Substituting into (41) gives

$$\nabla \times \nabla \times \mathbf{E} - k_0^2 \varepsilon_{rb}\mathbf{E} = k_0^2 \Delta\varepsilon_r(\mathbf{r})\mathbf{E}. \tag{42}$$

Physical interpretation. The left-hand side is the wave operator of the *homogeneous background medium*. The right-hand side acts as an *effective volume current source* induced by the permittivity contrast. Scattering is therefore viewed as radiation generated by this induced source.

5.2 Born approximation (weak, single scattering)

Write the total field as

$$\mathbf{E} = \mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{sc}},$$

where the incident field satisfies the background equation

$$\nabla \times \nabla \times \mathbf{E}_{\text{inc}} - k_0^2 \varepsilon_{rb}\mathbf{E}_{\text{inc}} = 0.$$

Subtracting from (42) yields the scattered-field equation

$$\nabla \times \nabla \times \mathbf{E}_{\text{sc}} - k_0^2 \varepsilon_{rb}\mathbf{E}_{\text{sc}} = k_0^2 \Delta\varepsilon_r(\mathbf{r})\mathbf{E}.$$

Born approximation assumption. If the permittivity contrast is weak and the object is small or low-index, then the total field inside the scatterer is well approximated by the incident field:

$$\mathbf{E} \approx \mathbf{E}_{\text{inc}}.$$

This yields the *first-order Born approximation*:

$$\nabla \times \nabla \times \mathbf{E}_{\text{sc}} - k_0^2 \varepsilon_{rb}\mathbf{E}_{\text{sc}} \approx k_0^2 \Delta\varepsilon_r(\mathbf{r})\mathbf{E}_{\text{inc}}(\mathbf{r}). \tag{43}$$

Scalar Green-function form. In a scalar approximation, this reduces to

$$(\nabla^2 + k_b^2) u_{\text{sc}}(\mathbf{r}) = -V(\mathbf{r}) u_{\text{inc}}(\mathbf{r}),$$

with solution

$$u_{\text{sc}}(\mathbf{r}) = \int V(\mathbf{r}') u_{\text{inc}}(\mathbf{r}') \frac{e^{ik_b|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}\, d^3 r'. \tag{44}$$

Validity. The Born approximation fails when multiple scattering, resonances, or strong index contrast occur.

5.3 Rayleigh scattering (dipole limit, $ka \ll 1$)

Consider a dielectric sphere of radius a and permittivity ε_p embedded in a background ε_b. If

$$ka \ll 1,$$

the electromagnetic field varies negligibly across the particle.

Key physical idea. The particle responds as an *induced electric dipole.* Higher multipoles are suppressed by powers of ka.

5.3.1 Quasi-static polarizability

Solving the electrostatic boundary-value problem yields the dipole polarizability

$$\alpha = 4\pi\varepsilon_0\varepsilon_{rb}a^3\left(\frac{\varepsilon_p - \varepsilon_b}{\varepsilon_p + 2\varepsilon_b}\right), \qquad \mathbf{p} = \alpha\mathbf{E}_{\text{inc}}(\mathbf{0}). \tag{45}$$

5.3.2 Rayleigh scattering cross section

The radiated power of an oscillating dipole gives the scattering cross section

$$\sigma_{\text{sc}} = \frac{k_b^4}{6\pi\varepsilon_0^2\varepsilon_{rb}^2}|\alpha|^2 = \frac{8\pi}{3}k_b^4a^6\left|\frac{\varepsilon_p - \varepsilon_b}{\varepsilon_p + 2\varepsilon_b}\right|^2. \tag{46}$$

Key consequence. Because $k_b = 2\pi/\lambda$, Rayleigh scattering scales as

$$\sigma_{\text{sc}} \propto \lambda^{-4},$$

explaining why short wavelengths scatter more strongly (e.g., blue sky).

5.4 Mie scattering (exact solution for a sphere)

For a sphere of arbitrary size, no approximation in ka is made. The fields are expanded in *vector spherical harmonics.*

Define:

$$x = k_b a, \qquad m = \frac{\tilde{n}_p}{\tilde{n}_b}.$$

Introduce Riccati–Bessel functions

$$\psi_\ell(z) = zj_\ell(z), \qquad \xi_\ell(z) = zh_\ell^{(1)}(z).$$

The Mie coefficients are

$$a_\ell = \frac{m\psi_\ell(mx)\psi'_\ell(x) - \psi_\ell(x)\psi'_\ell(mx)}{m\psi_\ell(mx)\xi'_\ell(x) - \xi_\ell(x)\psi'_\ell(mx)}, \tag{47}$$

$$b_\ell = \frac{\psi_\ell(mx)\psi'_\ell(x) - m\psi_\ell(x)\psi'_\ell(mx)}{\psi_\ell(mx)\xi'_\ell(x) - m\xi_\ell(x)\psi'_\ell(mx)}. \tag{48}$$

5.4.1 Cross sections

The total scattering and extinction cross sections are

$$\sigma_{\text{sc}} = \frac{2\pi}{k_b^2} \sum_{\ell=1}^{\infty} (2\ell+1) \left(|a_\ell|^2 + |b_\ell|^2 \right), \tag{49}$$

$$\sigma_{\text{ext}} = \frac{2\pi}{k_b^2} \sum_{\ell=1}^{\infty} (2\ell+1) \operatorname{Re}\{a_\ell + b_\ell\}, \tag{50}$$

with absorption

$$\sigma_{\text{abs}} = \sigma_{\text{ext}} - \sigma_{\text{sc}}.$$

Hierarchy connection.

- Rayleigh scattering emerges from Mie theory for $\ell = 1$ and $ka \ll 1$.
- Born approximation corresponds to neglecting multiple internal reflections.
- Mie theory captures resonances, morphology-dependent scattering, and near-field effects.

6 Pulse Propagation in Dispersive Media

An optical pulse is not a single frequency but a *superposition of monochromatic components*. In a dispersive medium, each frequency component accumulates a different phase during propagation, leading to pulse distortion and temporal broadening.

The goal of this section is to derive the pulse broadening law directly from Maxwell-consistent spectral propagation.

6.1 Pulse as a superposition of monochromatic components

Consider a linearly propagating pulse along the $+z$ direction in a homogeneous, linear, dispersive medium characterized by the propagation constant $k(\omega)$.

Using the analytic (positive-frequency) signal representation, the electric field can be written as

$$\tilde{E}(z,t) = \int_{-\infty}^{\infty} \tilde{E}(0,\omega) \exp\Big[\mathrm{i}k(\omega)z\Big] \exp\Big[-\mathrm{i}\omega t\Big] \,\mathrm{d}\omega. \tag{51}$$

Physical meaning. Each spectral component ω propagates independently and accumulates a phase $k(\omega)z$. Dispersion enters entirely through the frequency dependence of $k(\omega)$.

6.2 Carrier–envelope separation

Choose a carrier frequency ω_0 near the pulse center and define the detuning

$$\Omega = \omega - \omega_0.$$

Factor out the rapidly oscillating carrier:

$$\tilde{E}(z,t) = \exp\Big[\mathrm{i}(k_0 z - \omega_0 t)\Big] A(z,t), \qquad k_0 = k(\omega_0).$$

The slowly varying envelope $A(z,t)$ is then

$$A(z,t) = \int \tilde{A}(0,\Omega)\, \exp\Big\{\mathrm{i}[k(\omega_0+\Omega) - k_0]z\Big\}\, \exp\Big[-\mathrm{i}\Omega t\Big]\, \mathrm{d}\Omega. \tag{52}$$

Why this helps. Separating carrier and envelope allows us to track only the physically relevant *pulse shape evolution*, not the rapid optical oscillations.

6.3 Taylor expansion of the propagation constant

Because the pulse spectrum is typically narrow, $k(\omega)$ can be expanded around ω_0:

$$k(\omega_0+\Omega) = k_0 + k_1\Omega + \frac{1}{2}k_2\Omega^2 + \frac{1}{6}k_3\Omega^3 + \cdots, \tag{53}$$

where

$$k_m = \left.\frac{\mathrm{d}^m k}{\mathrm{d}\omega^m}\right|_{\omega_0}.$$

Interpretation of coefficients.

- $k_1 = 1/v_g$ determines the *group velocity.*
- k_2 describes *group-velocity dispersion (GVD).*
- Higher orders (k_3, etc.) describe pulse asymmetry and distortion.

6.4 Retarded time and the group-velocity frame

When a pulse propagates in a dispersive medium, two effects occur simultaneously: (i) the entire pulse envelope translates forward at the *group velocity*, and (ii) the pulse shape changes due to dispersion. To separate these two effects, it is convenient to describe the field in a reference frame that moves together with the pulse envelope.

Define the retarded (comoving) time variable

$$\tau = t - k_1 z, \qquad k_1 = \left.\frac{\mathrm{d}k}{\mathrm{d}\omega}\right|_{\omega_0} = \frac{1}{v_g}, \tag{54}$$

where v_g is the group velocity. Equivalently, $\tau = t - z/v_g$.

Physical meaning. The quantity z/v_g is the time required for the pulse envelope to travel a distance z. Thus τ measures time *relative to the pulse center*. In the (z,τ) frame, the pulse peak remains fixed near $\tau = 0$, and all remaining evolution corresponds to true pulse distortion rather than translation.

Role in the spectral representation. After expanding $k(\omega)$ about ω_0, the linear term produces the phase factor $\exp(\mathrm{i}k_1\Omega z)\exp(-\mathrm{i}\Omega t)$. Introducing τ combines these terms as

$$\exp(-\mathrm{i}\Omega t)\exp(\mathrm{i}k_1\Omega z) = \exp(-\mathrm{i}\Omega\tau),$$

which exactly removes the group-delay contribution from the integral.

Consequence. In the retarded-time frame, the envelope evolution is governed only by higher-order dispersion terms (k_2, k_3, ...). This leads directly to a Schrödinger-type propagation equation for the envelope, with z as the evolution variable and τ as the temporal coordinate.

6.5 Group-velocity dispersion (GVD) only

If higher-order dispersion is negligible, retain only k_2:

$$\tilde{A}(z,\Omega) = \tilde{A}(0,\Omega)\ \exp\left(\mathrm{i}\frac{k_2}{2}\Omega^2 z\right). \tag{55}$$

This expression shows that GVD introduces a *quadratic spectral phase*, which leads to temporal broadening.

6.6 Time-domain envelope equation

Using the Fourier-transform identity

$$\Omega^2 \longleftrightarrow -\frac{\partial^2}{\partial\tau^2},$$

the inverse transform of (55) yields

$$\frac{\partial A}{\partial z} = \mathrm{i}\frac{k_2}{2}\frac{\partial^2 A}{\partial\tau^2}. \tag{56}$$

Key observation. This equation is mathematically identical to a Schrödinger equation, with propagation distance z playing the role of time.

6.7 Gaussian pulse propagation and temporal broadening

Assume a Gaussian input pulse:

$$A(0,\tau) = A_0 \exp\left(-\frac{\tau^2}{2T_0^2}\right). \tag{57}$$

Its Fourier transform is

$$\tilde{A}(0,\Omega) = A_0\sqrt{2\pi}\,T_0 \exp\left(-\frac{\Omega^2 T_0^2}{2}\right).$$

After propagation under GVD:

$$\tilde{A}(z,\Omega) = A_0\sqrt{2\pi}\,T_0 \exp\left(-\frac{\Omega^2 T_0^2}{2}\right) \exp\left(\mathrm{i}\frac{k_2}{2}\Omega^2 z\right).$$

Inverting the transform gives

$$A(z,\tau) = A_0 \frac{T_0}{\sqrt{T_0^2 - \mathrm{i}k_2 z}} \exp\left[-\frac{\tau^2}{2(T_0^2 - \mathrm{i}k_2 z)}\right]. \tag{58}$$

6.8 Pulse broadening law

The intensity envelope remains Gaussian with width

$$T(z) = T_0\sqrt{1 + \left(\frac{k_2 z}{T_0^2}\right)^2}. \tag{59}$$

Define the *dispersion length*

$$L_D = \frac{T_0^2}{|k_2|}. \tag{60}$$

Then

$$T(z) = T_0\sqrt{1 + \left(\frac{z}{L_D}\right)^2}.$$

Physical meaning. After propagating one dispersion length, the pulse duration increases by a factor of $\sqrt{2}$.

6.9 Chirp induced by dispersion

Using

$$\frac{1}{T_0^2 - \mathrm{i}k_2 z} = \frac{T_0^2 + \mathrm{i}k_2 z}{T_0^4 + (k_2 z)^2},$$

the phase of (58) contains a term proportional to τ^2. This corresponds to a *linear frequency sweep* across the pulse (chirp), with instantaneous frequency increasing or decreasing depending on the sign of k_2.

PIC-oriented remarks (short but actionable)

- The TE/TM scalar reductions (20)–(23) are the exact algebraic bridge to slab waveguide eigenvalue problems.
- Dispersion parameters ($k_1, k_2, \ldots$ or $\beta_1, \beta_2, \ldots$) are waveguide-geometry dependent in PICs (material + waveguide dispersion).
- Scattering in PICs often uses Rayleigh-like scaling for sidewall roughness at small correlation lengths, and Born-like perturbation methods for weak index perturbations.

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