Lecture Notes: 2D (Slab) Dielectric Waveguides
From Maxwell to TE/TM Eigenvalue Equations, Dispersion, and Normalization

Contents

These notes derive the classical 2D slab (thin-film) waveguide starting from Maxwell's equations, leading to TE/TM scalar ODEs, guided-mode conditions, dispersion relations, normalized design curves, and mode normalization. The main reference is Osgood & Meng, *Principles of Photonic Integrated Circuits*, Ch. 3. (Source: Osgoodch3.pdf)

1 Geometry and Physical Picture

1.1 Three-layer slab waveguide (2D confinement)

We consider a *slab* waveguide that confines light only in one transverse direction. Let propagation be along $+z$ and confinement be in x. The structure has three layers:

- Cover (cladding) for $x > 0$: refractive index n_c
- Film (core) for $-d \le x \le 0$: refractive index n_f
- Substrate for $x < -d$: refractive index n_s

Assume typical ordering

$$n_c \le n_s < n_f.$$

This asymmetry is common in PICs: e.g., Si core on SiO_2 substrate with air/oxide cover.

1.2 Why guided modes exist (ray optics intuition)

A plane-wave component inside the film reflects at the top and bottom interfaces. If the incidence angle at an interface exceeds the critical angle, total internal reflection (TIR) occurs and the ray is trapped. For the film–cover and film–substrate interfaces define the critical angles:

$$\theta_c \equiv \sin^{-1}\left(\frac{n_c}{n_f}\right), \qquad \theta_s \equiv \sin^{-1}\left(\frac{n_s}{n_f}\right), \tag{1}$$

so $\theta_c \le \theta_s$. Guided rays require TIR at both interfaces, hence $\theta > \theta_s$.

Logic note. Ray optics predicts "trapping" but does not directly yield exact field profiles and polarization dependencies. For that, we use Maxwell + boundary conditions.

2 Maxwell Equations and the Helmholtz Equation in Layered Media

2.1 Time-harmonic Maxwell equations

Assume monochromatic time dependence

$$\mathbf{E}(\mathbf{r}, t) = \mathrm{Re}\{\mathbf{E}(\mathbf{r})\exp(-i\omega t)\}, \qquad \mathbf{H}(\mathbf{r}, t) = \mathrm{Re}\{\mathbf{H}(\mathbf{r})\exp(-i\omega t)\}.$$

In source-free, linear, nonmagnetic media (no free ρ, no free $\mathbf{J}$, and $\mu = \mu_0$), Maxwell's curl equations become (phasor form)

$$\nabla \times \mathbf{E} = -i\omega\mu_0\mathbf{H}, \tag{2}$$

$$\nabla \times \mathbf{H} = +i\omega\varepsilon_0 n^2(x)\mathbf{E}. \tag{3}$$

Here $n(x)$ is piecewise constant: $n = n_c$ for $x > 0$, $n = n_f$ for $-d \le x \le 0$, $n = n_s$ for $x < -d$.

2.2 Derive the vector Helmholtz equation

Take curl of (2):

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times (-i\omega\mu_0 \mathbf{H}) = -i\omega\mu_0(\nabla \times \mathbf{H}).$$

Now substitute (3) into the right-hand side:

$$-i\omega\mu_0(\nabla \times \mathbf{H}) = -i\omega\mu_0(i\omega\varepsilon_0 n^2 \mathbf{E}) = \omega^2\mu_0\varepsilon_0 n^2 \mathbf{E}.$$

Therefore

$$\nabla \times (\nabla \times \mathbf{E}) = \omega^2\mu_0\varepsilon_0 n^2(x)\mathbf{E}. \tag{4}$$

Use the vector identity

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}.$$

In each *uniform* layer (constant ε), Gauss law gives $\nabla \cdot \mathbf{D} = 0$ and $\mathbf{D} = \varepsilon \mathbf{E}$, so $\nabla \cdot \mathbf{E} = 0$ within that layer. Hence in each region:

$$\nabla(\nabla \cdot \mathbf{E}) = \mathbf{0}, \qquad \nabla \times (\nabla \times \mathbf{E}) = -\nabla^2 \mathbf{E}.$$

Then (4) becomes, layer-by-layer,

$$\nabla^2 \mathbf{E} + k_0^2 n^2 \mathbf{E} = 0, \qquad k_0 \equiv \frac{\omega}{c} = \omega\sqrt{\mu_0\varepsilon_0} = \frac{2\pi}{\lambda}. \tag{5}$$

This is the (vector) Helmholtz equation used throughout waveguide theory.

Logic note. Equation (5) holds *inside each uniform layer.* At interfaces, we must enforce Maxwell boundary conditions (tangential E and H continuity, etc.). The eigenvalue problem arises because only certain β allow all boundary conditions to be satisfied simultaneously.

3 Separation in z: Propagation Constant and Effective Index

3.1 Guided propagation along z

Because the structure is uniform in z, guided solutions are traveling waves along z:

$$\mathbf{E}(x, z) = \mathbf{E}(x)\exp(-i\beta z), \qquad \mathbf{H}(x, z) = \mathbf{H}(x)\exp(-i\beta z),$$

where β is the propagation constant of the mode.

Define the *effective index*

$$N_{\text{eff}} \equiv \frac{\beta}{k_0}. \tag{6}$$

Then the modal phase velocity along z is

$$v_p = \frac{\omega}{\beta} = \frac{c}{N_{\text{eff}}}.$$

3.2 Allowed range for guided modes (core-guided condition)

Inside the core, the bulk wavenumber magnitude is $k_0 n_f$. For a guided mode, β must satisfy

$$k_0 n_s < \beta < k_0 n_f \quad \Longleftrightarrow \quad n_s < N_{\text{eff}} < n_f. \tag{7}$$

Reason:

- If $\beta > k_0 n_f$, even the core cannot support propagation (transverse wavenumber imaginary everywhere).
- If $\beta < k_0 n_s$, the substrate supports propagating radiation away from the guide (radiation mode).

4 Polarization Decoupling: TE and TM in a Slab

4.1 2D slab assumption and field dependence

For a slab waveguide we assume invariance in y:

$$\frac{\partial}{\partial y} = 0.$$

Then TE and TM polarizations decouple and reduce Maxwell's vector problem to scalar ODEs.

Convention used here (consistent with many PIC texts).

- TE mode: $E_z = 0$ and the dominant component is $E_y(x)$.
- TM mode: $H_z = 0$ and the dominant component is $H_y(x)$.

4.2 Key outcome (what we will derive)

In each region (cover/core/substrate), both TE and TM satisfy the same *scalar* second-order ODE:

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + \left(k_0^2 n^2 - \beta^2\right)\Psi = 0, \tag{8}$$

where $\Psi = E_y$ (TE) or $\Psi = H_y$ (TM). The *difference* between TE and TM lies in the *boundary condition* at interfaces (tangential field continuity leads to different derivative constraints).

5 TE Guided Modes: Full Derivation (Fields, ODE, and Eigenvalue Equation)

5.1 Step 1: TE ansatz and reduction to a scalar ODE

Assume TE polarization:

$$\mathbf{E}(x,z) = \hat{\mathbf{y}}\, E_y(x)\exp(-i\beta z), \qquad E_x = E_z = 0.$$

Insert into the Helmholtz equation (5). Since only E_y exists, it satisfies

$$\nabla^2 E_y + k_0^2 n^2 E_y = 0.$$

Now compute ∇^2 in (x,z) (no y dependence):

$$\nabla^2 E_y = \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2}.$$

But $E_y(x,z) = E_y(x)\exp(-i\beta z)$, hence

$$\frac{\partial^2 E_y}{\partial z^2} = \frac{\partial^2}{\partial z^2}\left(E_y(x)e^{-i\beta z}\right) = (-i\beta)^2 E_y(x)e^{-i\beta z} = -\beta^2 E_y(x)e^{-i\beta z}.$$

Therefore

$$\left(\frac{\mathrm{d}^2 E_y}{\mathrm{d}x^2} - \beta^2 E_y + k_0^2 n^2 E_y\right)e^{-i\beta z} = 0$$

and canceling the nonzero factor $e^{-i\beta z}$ gives the TE scalar ODE:

$$\frac{\mathrm{d}^2 E_y}{\mathrm{d}x^2} + \left(k_0^2 n^2 - \beta^2\right)E_y = 0, \qquad \text{(TE)} \tag{9}$$

valid in each layer with its local n.

5.2 Step 2: Guided-mode form of the solution (oscillatory in core, evanescent outside)

For a guided mode, the field is oscillatory in the core and decays into cover/substrate. Define the transverse constants (cover/core/substrate):

$$\delta^2 \equiv \beta^2 - k_0^2 n_c^2 \quad (x > 0), \tag{10}$$
$$\kappa^2 \equiv k_0^2 n_f^2 - \beta^2 \quad (-d \le x \le 0), \tag{11}$$
$$\gamma^2 \equiv \beta^2 - k_0^2 n_s^2 \quad (x < -d). \tag{12}$$

For guided modes, (7) implies $\beta > k_0 n_s \ge k_0 n_c$ so $\delta, \gamma > 0$, and also $\beta < k_0 n_f$ so $\kappa > 0$.

Now write the general TE solutions region-by-region:

Cover ($x > 0$): decaying exponential.

$$E_y(x) = Ce^{-\delta x}.$$

Core ($-d \le x \le 0$): sinusoidal.

$$E_y(x) = A\cos(\kappa x) + B\sin(\kappa x).$$

Substrate ($x < -d$): decaying exponential away from the core. To ensure decay as $x \to -\infty$, write it as

$$E_y(x) = De^{+\gamma(x+d)} \quad (x \le -d),$$

because when $x \to -\infty$, $(x+d) \to -\infty$ and $e^{\gamma(x+d)} \to 0$.

Thus:

$$E_y(x) = \begin{cases} Ce^{-\delta x}, & x \ge 0, \\ A\cos(\kappa x) + B\sin(\kappa x), & -d \le x \le 0, \\ De^{\gamma(x+d)}, & x \le -d. \end{cases} \tag{13}$$

5.3 Step 3: TE boundary conditions and why they reduce to E_y and $\partial_x E_y$

At a dielectric interface with no surface charge/current:

$$\mathbf{E}_t \text{ continuous}, \qquad \mathbf{H}_t \text{ continuous}.$$

For TE, E_y is tangential everywhere, so E_y must be continuous.

To convert $\mathbf{H}_t$ continuity into a condition on $\partial_x E_y$, compute $\mathbf{H}$ from Maxwell:

$$\nabla \times \mathbf{E} = -i\omega\mu_0 \mathbf{H}.$$

With $\mathbf{E} = \hat{\mathbf{y}} E_y(x) e^{-i\beta z}$, the curl is

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} = -\hat{\mathbf{x}}\frac{\partial E_y}{\partial z} + \hat{\mathbf{z}}\frac{\partial E_y}{\partial x}.$$

Since $\frac{\partial E_y}{\partial z} = -i\beta E_y$,

$$\nabla \times \mathbf{E} = -\hat{\mathbf{x}}(-i\beta E_y) + \hat{\mathbf{z}}\frac{\mathrm{d}E_y}{\mathrm{d}x} = i\beta E_y\,\hat{\mathbf{x}} + \left(\frac{\mathrm{d}E_y}{\mathrm{d}x}\right)\hat{\mathbf{z}}.$$

Therefore

$$-i\omega\mu_0 \mathbf{H} = i\beta E_y\,\hat{\mathbf{x}} + \left(\frac{\mathrm{d}E_y}{\mathrm{d}x}\right)\hat{\mathbf{z}},$$

so

$$\mathbf{H} = \frac{1}{\omega\mu_0}\left(-\beta E_y\,\hat{\mathbf{x}} + i\frac{\mathrm{d}E_y}{\mathrm{d}x}\,\hat{\mathbf{z}}\right).$$

The tangential magnetic field at an $x = \text{const}$ interface includes H_z, and $H_z \propto \frac{\mathrm{d}E_y}{\mathrm{d}x}$. Thus H_z continuity implies $\frac{\mathrm{d}E_y}{\mathrm{d}x}$ continuity (here μ is the same).

Hence TE interface conditions are:

$$E_y \text{ continuous}, \qquad \frac{\mathrm{d}E_y}{\mathrm{d}x} \text{ continuous}. \tag{14}$$

Logic note. This is the key reason TE and TM differ: TM ends up with a weighted derivative condition involving $1/n^2$.

5.4 Step 4: Apply boundary conditions at $x = 0$

Continuity of E_y:

$$E_y(0^+) = E_y(0^-) \quad \Rightarrow \quad C = A\cos(0) + B\sin(0) = A.$$

So $C = A$.

Continuity of derivative:

$$\left.\frac{\mathrm{d}E_y}{\mathrm{d}x}\right|_{0^+} = \left.\frac{\mathrm{d}E_y}{\mathrm{d}x}\right|_{0^-}.$$

Compute derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(Ce^{-\delta x}\right) = -\delta Ce^{-\delta x} \Rightarrow \left.\frac{\mathrm{d}E_y}{\mathrm{d}x}\right|_{0^+} = -\delta C = -\delta A.$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(A\cos(\kappa x) + B\sin(\kappa x)\right) = -A\kappa\sin(\kappa x) + B\kappa\cos(\kappa x) \Rightarrow \left.\frac{\mathrm{d}E_y}{\mathrm{d}x}\right|_{0^-} = B\kappa.$$

Thus

$$-\delta A = \kappa B \quad \Rightarrow \quad B = -\frac{\delta}{\kappa}A. \tag{15}$$

5.5 Step 5: Apply boundary conditions at $x = -d$

First continuity of E_y at $x = -d$:

$$E_y(-d^+) = E_y(-d^-).$$

Core side:

$$E_y(-d^+) = A\cos(-\kappa d) + B\sin(-\kappa d) = A\cos(\kappa d) - B\sin(\kappa d).$$

Substrate side:

$$E_y(-d^-) = De^{\gamma((-d)+d)} = D.$$

So

$$D = A\cos(\kappa d) - B\sin(\kappa d). \tag{16}$$

Now continuity of derivative at $x = -d$:

$$\left.\frac{\mathrm{d}E_y}{\mathrm{d}x}\right|_{-d^+} = \left.\frac{\mathrm{d}E_y}{\mathrm{d}x}\right|_{-d^-}.$$

Core derivative at $x = -d$:

$$\frac{\mathrm{d}E_y}{\mathrm{d}x} = -A\kappa\sin(\kappa x) + B\kappa\cos(\kappa x) \Rightarrow \left.\frac{\mathrm{d}E_y}{\mathrm{d}x}\right|_{-d^+} = -A\kappa\sin(-\kappa d) + B\kappa\cos(-\kappa d) = A\kappa\sin(\kappa d) + B\kappa\cos(\kappa d).$$

Substrate derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(De^{\gamma(x+d)}\right) = \gamma De^{\gamma(x+d)} \Rightarrow \left.\frac{\mathrm{d}E_y}{\mathrm{d}x}\right|_{-d^-} = \gamma D.$$

Hence

$$A\kappa\sin(\kappa d) + B\kappa\cos(\kappa d) = \gamma D. \tag{17}$$

Substitute (16) for D:

$$A\kappa\sin(\kappa d) + B\kappa\cos(\kappa d) = \gamma\left(A\cos(\kappa d) - B\sin(\kappa d)\right).$$

Now substitute $B = -(\delta/\kappa)A$ from (15), divide by $A \neq 0$:

$$\kappa\sin(\kappa d) - \delta\cos(\kappa d) = \gamma\cos(\kappa d) + \gamma\frac{\delta}{\kappa}\sin(\kappa d).$$

Collect sin and cos terms:

$$\left(\kappa - \frac{\gamma\delta}{\kappa}\right)\sin(\kappa d) = (\gamma + \delta)\cos(\kappa d).$$

Divide by $\cos(\kappa d)$:

$$\tan(\kappa d) = \frac{\kappa(\gamma+\delta)}{\kappa^2 - \gamma\delta}. \tag{18}$$

This is the TE guided-mode eigenvalue equation (as in the source chapter).

Logic note. Equation (18) is a *transcendental* condition: only discrete β satisfy it. Each solution corresponds to one guided mode (TE_0, TE_1, ...).

6 TM Guided Modes: What Changes (ODE same, boundary condition different)

6.1 TM scalar ODE

For TM, the standard approach is to choose $H_y(x)$ as the scalar unknown (the field parallel to interfaces). One again obtains in each region:

$$\frac{\mathrm{d}^2 H_y}{\mathrm{d}x^2} + \left(k_0^2 n^2 - \beta^2\right)H_y = 0, \qquad \text{(TM)} \tag{19}$$

6.2 TM boundary conditions (the key difference)

At interfaces, tangential H is continuous so H_y is continuous.

The second condition comes from continuity of tangential E, which for TM involves E_z. Using Maxwell, one finds (within each uniform layer) that

$$E_z \propto \frac{1}{\varepsilon}\frac{\mathrm{d}H_y}{\mathrm{d}x} = \frac{1}{\varepsilon_0 n^2}\frac{\mathrm{d}H_y}{\mathrm{d}x}.$$

Therefore TM interface conditions are:

$$H_y \text{ continuous}, \qquad \frac{1}{n^2}\frac{\mathrm{d}H_y}{\mathrm{d}x} \text{ continuous}. \tag{20}$$

Logic note. TE "derivative continuity" is unweighted because μ is constant. TM involves ε, so the derivative condition is weighted by $1/n^2$. This is why TE and TM dispersion differ in asymmetric guides.

7 Normalized Design Parameters: V, a, b, Cutoff

7.1 Normalized frequency (thickness parameter)

Define the transverse wavenumber in the core:

$$\kappa^2 = k_0^2 n_f^2 - \beta^2.$$

A key dimensionless parameter is the "normalized frequency" (often called V):

$$V \equiv k_0 d \sqrt{n_f^2 - n_s^2}. \tag{21}$$

Increasing d, increasing index contrast, or decreasing wavelength increases V and increases mode count.

7.2 Asymmetry parameter

Define

$$a \equiv \frac{n_s^2 - n_c^2}{n_f^2 - n_s^2}. \tag{22}$$

If $n_c = n_s$ (symmetric slab), then $a = 0$.

7.3 Normalized propagation constant

Define

$$b \equiv \frac{N_{\text{eff}}^2 - n_s^2}{n_f^2 - n_s^2} = \frac{(\beta/k_0)^2 - n_s^2}{n_f^2 - n_s^2}. \tag{23}$$

Note the bounds for guided modes:

$$n_s < N_{\text{eff}} < n_f \quad \Rightarrow \quad 0 < b < 1.$$

7.4 Normalized dispersion equation (TE)

The eigenvalue equation can be rewritten into a normalized form $b = b(V)$. (Exact algebra is tedious but follows by expressing κ, γ, δ in terms of b, V, a.) A commonly used normalized TE form is

$$V\sqrt{1-b} = m\pi + \tan^{-1}\left(\sqrt{\frac{b}{1-b}}\right) + \tan^{-1}\left(\sqrt{\frac{b+a}{1-b}}\right), \qquad m = 0, 1, 2, \ldots \tag{24}$$

which allows universal design charts.

7.5 Cutoff condition

A mode reaches cutoff when it becomes non-guided, i.e. when $b \to 0$ (equivalently $N_{\text{eff}} \to n_s$). At $b = 0$, (24) gives the cutoff normalized frequency:

$$V_c^{(m)} = m\pi + \tan^{-1}(\sqrt{a}). \tag{25}$$

For a symmetric slab $a = 0$, so $V_c^{(m)} = m\pi$, and in particular $m = 0$ has $V_c^{(0)} = 0$, meaning TE_0 exists for any thickness in the symmetric case.

Logic note. The "no-cutoff" of the fundamental mode in symmetric guides is the waveguide analog of a ground state: there is always at least one bound solution if the potential (index step) is symmetric and supports confinement.

8 Symmetric Slab: Even/Odd Mode Families

Assume symmetric cladding/substrate: $n_c = n_s$, so $\delta = \gamma$ and $a = 0$. Then eigenvalue conditions separate into even (symmetric) and odd (antisymmetric) core fields.

Shifting origin to the slab center simplifies the algebra (not repeated here). The two families become:

$$\tan\left(\frac{\kappa d}{2}\right) = \frac{\gamma}{\kappa} \qquad \text{(even modes)}, \tag{26}$$

$$\tan\left(\frac{\kappa d}{2}\right) = \frac{\kappa}{\gamma} \qquad \text{(odd modes)}. \tag{27}$$

Logic note. Even/odd arises because in a symmetric structure the boundary conditions are invariant under $x \to -x$, so solutions can be chosen with definite parity.

Electric field distributions for symmetric slab (TE modes)

Assume a symmetric slab waveguide with

$$n_c = n_s, \qquad \delta = \gamma,$$

and choose the origin at the center of the core. The core then occupies $|x| \le d/2$.

For TE modes, the dominant field component is $E_y(x)$, which satisfies

$$\frac{\mathrm{d}^2 E_y}{\mathrm{d}x^2} + \left(k_0^2 n^2 - \beta^2\right) E_y = 0$$

in each region.

Because the structure is symmetric under $x \to -x$, the solutions separate into *even (symmetric)* and *odd (antisymmetric)* mode families.

Even TE modes (symmetric about $x = 0$). In the core region ($|x| \le d/2$), the field is cosine-like:

$$E_y(x) = A\cos(\kappa x), \qquad |x| \le \frac{d}{2}. \tag{28}$$

In the cladding regions ($|x| > d/2$), the field decays exponentially:

$$E_y(x) = A\cos\left(\frac{\kappa d}{2}\right)\exp\left[-\delta\left(|x| - \frac{d}{2}\right)\right], \qquad |x| > \frac{d}{2}. \tag{29}$$

The corresponding dispersion relation is

$$\tan\left(\frac{\kappa d}{2}\right) = \frac{\delta}{\kappa}.$$

Odd TE modes (antisymmetric about $x = 0$). In the core region ($|x| \le d/2$), the field is sine-like:

$$E_y(x) = A\sin(\kappa x), \qquad |x| \le \frac{d}{2}. \tag{30}$$

In the cladding regions ($|x| > d/2$), the field decays exponentially with a sign change:

$$E_y(x) = A\,\mathrm{sgn}(x)\,\sin\left(\frac{\kappa d}{2}\right)\exp\left[-\delta\left(|x| - \frac{d}{2}\right)\right], \qquad |x| > \frac{d}{2}. \tag{31}$$

The corresponding dispersion relation is

$$\tan\left(\frac{\kappa d}{2}\right) = \frac{\kappa}{\delta}.$$

Physical interpretation.

- Even modes have maximum electric field at the center of the slab ($x = 0$).
- Odd modes have a node at the center and opposite field polarity on either side.
- Higher-order modes correspond to increasing numbers of transverse field oscillations.
- The exponential tails represent evanescent penetration into the claddings, which increases as the mode approaches cutoff.

9 Mode Normalization, Power, and Effective Width

9.1 Time-average power carried by a guided mode

The time-average Poynting vector is

$$\langle \mathbf{S} \rangle = \frac{1}{2} \operatorname{Re}\{\mathbf{E} \times \mathbf{H}^*\}.$$

Power flow is the z-component integrated over transverse coordinate:

$$P = \int_{-\infty}^{\infty} \langle S_z \rangle \mathrm{d}x. \tag{32}$$

For TE modes, one can express H_x in terms of E_y using Maxwell and show (after simplification) that

$$P = \frac{\beta}{2\omega\mu_0} \int_{-\infty}^{\infty} |E_y(x)|^2 \mathrm{d}x. \tag{33}$$

This formula is extremely useful: it normalizes the arbitrary amplitude constant A in the mode profile.

9.2 Effective mode width

Because the TE mode decays with characteristic lengths $1/\delta$ and $1/\gamma$ outside the core, the field extends beyond d. A practical measure is the effective transverse extent

$$T_{\text{eff}} \equiv d + \frac{1}{\delta} + \frac{1}{\gamma}. \tag{34}$$

Logic note. As the core becomes very thin (or wavelength increases), confinement weakens (δ, γ decrease), so $1/\delta$ and $1/\gamma$ grow and the mode spreads—this is why tapering helps fiber-to-chip coupling.

10 Waveguide Dispersion and Group Velocity

10.1 Group velocity and group index

For guided waves, the relevant dispersion is $\beta(\omega)$, not $k(\omega)$. Define group velocity:

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}\beta}. \tag{35}$$

Define group index:

$$N_g \equiv \frac{c}{v_g} = c\,\frac{\mathrm{d}\beta}{\mathrm{d}\omega}. \tag{36}$$

Logic note. Even if the material index were constant, N_{eff} depends on wavelength because confinement changes with λ. This is *waveguide dispersion* (geometry-induced), important for PIC delay lines and routers.

11 Radiation Modes (Completeness idea)

Guided modes are discrete (countable), but a general field requires a complete basis. The remaining solutions form a continuous set of *radiation modes* where at least one cladding region supports propagation (e.g. $\beta < k_0 n_s$ gives substrate radiation). These modes are essential for:

- describing scattering and roughness loss,
- coupling from free space or substrate into the guide,
- completeness expansions of arbitrary initial fields.

12 Graded-Index Slab (Diffused) Waveguides: Ray-Optics Quantization

When the index varies smoothly $n = n(x)$ (diffusion/implantation), the slab is no longer abrupt. A physically insightful method uses ray optics with a turning point x_t where $n(x_t) = N_{\text{eff}}$. A quantization condition is obtained by requiring the total phase accumulation in one bounce cycle to equal $2\pi m$.

12.1 Phase accumulation integral

Transverse phase increment in a small step $\mathrm{d}x$ is

$$\mathrm{d}\phi = k_0 \sqrt{n^2(x) - N_{\text{eff}}^2}\,\mathrm{d}x.$$

Over a complete trajectory (down and up) one gets

$$\Phi = 2k_0 \int_0^{x_t} \sqrt{n^2(x) - N_{\text{eff}}^2}\,\mathrm{d}x + (\text{interface phase terms}). \tag{37}$$

Then the mode condition is $\Phi = 2\pi m$ with specific interface phase offsets depending on the turning/reflection behavior.

Logic note. This is the waveguide analog of WKB quantization: the "bound state" exists when the optical phase closes on itself after one transverse cycle.

13 Summary Checklist (what you should take to 3D waveguides)

- Start from Maxwell $\Rightarrow$ Helmholtz in each layer.
- Use $e^{-i\beta z}$ to reduce PDE $\Rightarrow$ transverse ODE.
- Guided modes: oscillatory in core (κ real), evanescent outside (δ, γ real).
- TE: E_y continuous and $\frac{\mathrm{d}E_y}{\mathrm{d}x}$ continuous.
- TM: H_y continuous and $\frac{1}{n^2}\frac{\mathrm{d}H_y}{\mathrm{d}x}$ continuous.

- Dispersion relation is a transcendental eigenvalue equation $\Rightarrow$ discrete modes.
- Normalized parameters V, a, b give universal design curves and cutoff.
- Group velocity uses $\beta(\omega)$: $v_g = \mathrm{d}\omega/\mathrm{d}\beta$ includes waveguide dispersion.

End of 2D slab waveguide lecture note

14 Three-Dimensional Waveguide Structures and the Need for Numerical Methods

So far, we have focused on planar (1D) and slab (2D) waveguides, where the material variation occurs along a single transverse coordinate and exact analytical solutions are possible. Most practical photonic devices, however, rely on fully three-dimensional (3D) waveguide geometries, for which closed-form solutions generally do not exist.

This section classifies common 3D waveguide structures, explains why analytical methods fail, and introduces numerical and semi-analytical approaches, beginning with the effective index method.

14.1 Common 3D dielectric waveguide geometries

In a 3D waveguide, the refractive index varies in *two* transverse directions. Typical structures include:

- **Rectangular (channel) waveguide**: A high-index rectangular core surrounded by lower-index cladding.
- **Rib waveguide**: A partially etched slab, widely used in integrated photonics for fabrication tolerance.
- **Strip (ridge) waveguide**: A fully etched high-index strip on a lower-index substrate.
- **Buried channel waveguide**: A core embedded within cladding on all sides.
- **Photonic wire**: A high-index-contrast nanowire with strong subwavelength confinement.
- **Slot waveguide**: Two high-index rails separated by a low-index gap, concentrating the electric field in the slot.
- **Circular fiber-like waveguides**: Cylindrical symmetry, but finite boundaries in both transverse directions.

In all these cases, the refractive index depends on both transverse coordinates:

$$\varepsilon = \varepsilon(x, y), \qquad \mu \approx \mu_0.$$

Key observation. Once the structure varies in both x and y, the separability that enabled exact slab solutions is generally lost.

14.2 Why analytical solutions fail in 3D waveguides

Assume monochromatic propagation along z with propagation constant β:

$$\mathbf{E}(x, y, z) = \mathbf{E}(x, y)\, e^{i\beta z}, \qquad \mathbf{H}(x, y, z) = \mathbf{H}(x, y)\, e^{i\beta z}.$$

Inserting this into Maxwell's equations yields coupled partial differential equations (PDEs) in x and y for the transverse and longitudinal field components.

Loss of separability

In slab waveguides, the refractive index depends only on one coordinate, e.g. $\varepsilon(x)$. This allows separation of variables:

$$E_y(x,z) = E_y(x)\, e^{i\beta z},$$

leading to a scalar ODE in x.

In contrast, for 3D waveguides:

$$\varepsilon = \varepsilon(x,y),$$

and the governing equation becomes:

$$\nabla_\perp^2 \Psi(x,y) + \left(k_0^2 \varepsilon(x,y) - \beta^2\right) \Psi(x,y) = 0,$$

where Ψ represents a transverse field component.

This is a **2D eigenvalue problem**, not reducible to independent 1D equations.

Logical consequence.

- No coordinate transformation exists that diagonalizes the operator in general.
- Boundary conditions couple x and y behavior.
- TE/TM separation breaks down: modes are *hybrid*.

Therefore, closed-form solutions using elementary functions do not exist except for highly symmetric cases.

14.3 Hybrid modes in 3D waveguides

In 3D structures, neither E_z nor H_z vanishes:

$$E_z \neq 0, \qquad H_z \neq 0.$$

Modes are classified as:

- **Quasi-TE**: dominant transverse electric field component
- **Quasi-TM**: dominant transverse magnetic field component

These labels are approximate and become exact only in the slab or weakly guiding limits.

Physical reason. Strong transverse confinement requires field curvature in both x and y, which inevitably generates longitudinal field components through Maxwell's curl equations.

14.4 Numerical eigenvalue problem formulation

The 3D waveguide problem is fundamentally an eigenvalue problem:

$$\mathcal{L}[\mathbf{E}(x,y)] = \beta^2\, \mathbf{E}(x,y),$$

where $\mathcal{L}$ is a vector differential operator involving spatially varying $\varepsilon(x,y)$.

The task is to compute:

- Eigenvalues β_m (effective indices $n_{\text{eff},m} = \beta_m/k_0$)
- Corresponding mode profiles $\mathbf{E}_m(x,y)$

This requires numerical discretization.

14.5 Effective Index Method (EIM)

The effective index method is a *semi-analytical* approximation that reduces a 3D problem to a sequence of 2D (slab) problems.

Basic idea

1. Solve a slab waveguide problem in one transverse direction (e.g. x), assuming infinite extent in the other (y).
2. Extract an effective refractive index $n_{\text{eff}}(x)$.
3. Use this effective index profile as the refractive index in a second slab problem along the orthogonal direction (y).

Logic. The strongest confinement is typically in one direction. If that confinement is treated exactly, the remaining direction can be treated approximately.

Mathematical structure

Step 1 (vertical slab):

$$\frac{d^2\Psi}{dx^2} + \left(k_0^2 n^2(x) - \beta_x^2\right)\Psi = 0 \quad \Rightarrow \quad n_{\text{eff}}^{(x)} = \beta_x / k_0.$$

Step 2 (lateral slab):

$$\frac{d^2\Phi}{dy^2} + \left(k_0^2 [n_{\text{eff}}^{(x)}(y)]^2 - \beta^2\right)\Phi = 0.$$

Result. An approximate propagation constant β and mode profile.

Accuracy and limitations

- Accurate for weakly guiding and high-aspect-ratio waveguides
- Fails for strong confinement and high-index contrast
- Cannot capture full vectorial coupling effects

14.6 Relation to commercial solvers (e.g. Lumerical)

The effective index method is conceptually related to—but not identical with— the mode solvers used in tools such as Lumerical MODE.

- **EIM**: sequential slab approximations (semi-analytical)
- **Lumerical MODE**: full-vector numerical eigenmode solver (finite-difference or finite-element discretization)

Lumerical directly solves the 2D vector eigenvalue problem:

$$\nabla \times \left(\mu^{-1} \nabla \times \mathbf{E}\right) - k_0^2 \varepsilon(x, y) \mathbf{E} = 0,$$

without assuming separability or dominant field components.

Key distinction.

- EIM is fast, intuitive, and approximate
- Numerical solvers are accurate, general, and computationally intensive

Summary

- 3D waveguides are essential for integrated photonics
- Analytical solutions fail due to loss of separability and hybridization
- The problem becomes a 2D vector eigenvalue problem
- Effective index method provides physical intuition and fast estimates
- Full-vector numerical solvers are required for accurate design